



What is reconstruction for ordered sets?

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Abstract

Reconstruction questions arise when studying interactions between the isomorphic type of a structure and the isomorphic types of its substructures. In this survey paper we are interested in binary relations and particularly we focus on partially ordered binary relations. We present most of the known results on partially ordered sets and that for different kinds of reconstruction: among them we have the Fraïssé-reconstruction, the Ulam-reconstruction, the max-reconstruction and the set-reconstruction. © 2004 Elsevier B.V. All rights reserved.

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1. Introduction

In this paper, our main interest is in reconstruction questions on finite partially ordered sets, orders for short. However, before concentrating our attention on orders, we give a general overview of reconstruction results for other binary relations. Also we give some references to survey papers dedicated to these topics. The presentation we choose, for the results that we know about the reconstruction of orders, focuses on constructive approaches. Our presentation is of course not exhaustive, rather we give, as often as possible, some hints of proof for the results that we present. The elements of the proofs that we choose are those that we think important to understand the proof techniques which are used. For complementary details, the reader can refer to the original papers. All through the survey paper, unless otherwise stated, we are only concerned with finite sets.

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1.1. General context

The reconstruction problems that we discuss take place in the study of relations. More particularly, they enter into the morphologic approach to relations: that is, the study of the behavior of relations together with their sub-relations. Two standard examples of the morphologic approach are, the definition of classes of relations by requiring or by forbidding sub-relations, and the reconstruction problems we discuss in the sequel. First recall the notion of relation: given a set E and an integer k , a k -ary relation (or relation with arity k), on a ground set E , is a mapping from the set of k -tuples induced by E , into a 2-element set. Therefore, a k -ary relation is a subset of the Cartesian product E^k . A relation is finite as soon as its ground set is finite. Thus, unless otherwise stated, we are only concerned with finite relations.

Given a relation on an n -element ground set, one can ask whether it is completely determined by the knowledge of its sub-relations. We can refine this question by asking which subsets, or which families, of sub-relations are sufficient to characterize the initial relation. All that we need to specify is what knowledge of the sub-relations we assume, and also, in what characterizations we are interested. For example, given a relation R on an n -element ground set E , we can ask for exact knowledge, that is for equality, both for the relation and for the sub-relations. Also, we can ask for structural knowledge, that is the isomorphism type, of the sub-relations, and then for the isomorphism type of the relation. Under these hypotheses assume that we are given:

- with exact knowledge (i.e., equality):
 - $\mathcal{E}K_2(R) = \{R[X] : |X| = 2 \text{ and } X \subset E\}$, the set of the 2-element sub-relations.
 - $\mathcal{E}K_{n-1}(R) = \{R[X] : |X| = n - 1 \text{ and } X \subset E\}$, the set of the $(n - 1)$ -element sub-relations.
- with structural knowledge (i.e., the isomorphism type):
 - $\mathcal{S}K_2(R) = \{\widehat{R[X]} : |X| = 2 \text{ and } X \subset E\}$, the set of the isomorphism types of the 2-element sub-relations.
 - $\mathcal{S}K_{n-1}(R) = \{\widehat{R[X]} : |X| = n - 1 \text{ and } X \subset E\}$, the set of the isomorphism types of the $(n - 1)$ -element sub-relations.
 - $\mathcal{F}SK_2(R) = (\widehat{R[X]})_{\{X: |X|=2 \text{ and } X \subset E\}}$, the family of the isomorphism types of the 2-element sub-relations.
 - $\mathcal{F}SK_{n-1}(R) = (\widehat{R[X]})_{\{X: |X|=n-1 \text{ and } X \subset E\}}$, the family of the isomorphism types of the $(n - 1)$ -element sub-relations.

The relation R is completely determined by the set $\mathcal{E}K_2(R)$, and also by the set $\mathcal{E}K_{n-1}(R)$. Indeed, we have $R = \bigcup_{Z \in \mathcal{E}K_2(R)} Z = \bigcup_{Z \in \mathcal{E}K_{n-1}(R)} Z$. The relation R cannot be determined, either from the set $\mathcal{S}K_2(R)$ or from the family $\mathcal{F}SK_2(R)$. Indeed, for any relation S having the same number of elements as R , we have $\mathcal{S}K_2(R) = \mathcal{S}K_2(S)$ and $\mathcal{F}SK_2(R) = \mathcal{F}SK_2(S)$. For the set $\mathcal{S}K_{n-1}(R)$, or for the family $\mathcal{F}SK_{n-1}(R)$, things become much more difficult. Reconstruction questions are concerned with this latter kind of problem.

1.2. The Stanislaw Marcin Ulam question

In 1973, Harary's survey of the reconstruction conjecture recounts the origins of the problem [24]:

The author first heard of this fascinating problem when Kelly [30] proved the theorem for trees in 1957. This result was obtained in Kelly's doctoral dissertation which was written under Ulam, who published in [58] a statement of the problem in 1960 (although it was already known to him in 1929, when he assiduously collected mathematical problems posed by his fellow graduate students and professors in Lwów, Poland.)

...

In Ulam [58, p. 29], the following statement appears: "Algebraic Problem 1: Suppose that in two sets A and B , each of n elements, there is a defined distance function ρ for every pair of distinct points, with values either 1 or 2, and $\rho(p, p) = 0$. Assume that for every subset of $n - 1$ points of A , there exists an isometric system of $n - 1$ points of B , and that the number of distinct subsets isometric to any given subset of $n - 1$ points is the same in A as in B . Are A and B isometric?"

Therefore Harary proposed to call the problem the reconstruction conjecture, nowadays better known as the graph (vertex) reconstruction conjecture. The reconstruction conjecture states: *Any graph, with at least three vertices, can be reconstructed up to isomorphism from its collection of one-vertex-deleted subgraphs.* In fact, earlier in 1964, Harary gave a nice and useful reformulation of the graph reconstruction question [23]: *Somebody draws on cards all one-vertex-deleted subgraphs of an unknown graph G , one subgraph per card. Can we reconstruct the original graph from this deck of cards, up to isomorphism?*

1.3. Ulam-type reconstruction

Due to the difficulty of Ulam's question, other types of reconstruction were introduced. We discuss some of them in the following.

1.3.1. Binary relations

The original question, which is stated in terms of isometric sets, is then equivalent to the reconstruction question on symmetric binary relations. When the binary relations are no longer assumed to be symmetric we get a more general framework: *Can any binary relation, with at least three elements, be reconstructed—up to isomorphism—from its family of one-element-deleted sub-relations?*

Three elements are necessary since, for the P_2 and the $\overline{P_2}$ loopless graphs (symmetric and irreflexive relations), every one-element deleted sub-graph is isomorphic to the loopless graph on one element. Particular instances of this general question can then be created simply by changing "Can any binary relation with at least three elements ..." to "Can any binary relation fulfilling property Π and on at least α_Π elements...".

1.3.2. Set-reconstruction

Instead of the family of the isomorphism type of the one-element-deleted sub-relations, we are interested in the set. That is, the number of sub-relations having a given isomorphism

type is no longer known. The set-reconstruction question is then: *Can any binary relation, with at least four elements, be reconstructed—up to isomorphism—from its set of one-element-deleted sub-relations?*

Four elements are necessary since the P_3 and the $\overline{P_3}$ loopless graphs have the same set of isomorphism types of 2-element sub-relations. Note that the P_3 and the $K_2 \oplus K_1$ graphs also have the same set of isomorphism types, taken over their 2-element sub-relations. Some remarks about set-reconstruction can be found in the survey paper of Capobianco and Molluzzo [7]. Regarding orders, it seems that the only work has been done by Das [8,10–12] from 1973 to 1981.

1.3.3. Sub-families

We can restrict the given family to be taken over some particular elements, that is, over elements fulfilling a given property. At this point, we have to notice that, given two different relations, two different properties can lead to the same sub-family. The sub-family reconstruction question is then: *Can any binary relation, with at least α_Π elements, be reconstructed—up to isomorphism—from the family of one-element-deleted sub-relations obtained when the deleted elements fulfill the property Π ?*

This kind of approach has been followed for graphs. See for example the end-vertex case on p. 237 of the survey paper of Bondy and Hemminger [6]. Of course, one can ask for the set of one-element-deleted sub-relations instead of the family. Regarding orders, by taking for Π the property “being a maximal element”, we obtain a question addressed by Sands [48] in 1985.

1.4. The Roland Fraïssé question

Two relations R and R' on the same ground set E are said to be *I-hypomorphic* provided that for $X \subseteq E$, if $|X| \in I$, respectively $-|X| \in I$, then $R[X] \simeq R'[X]$, respectively $R - X \simeq R' - X$. A relation R is then said to be *I-reconstructible* provided that for every relation R' , if R and R' are *I-hypomorphic*, then R and R' are isomorphic. On the one hand, the $\{1, \dots, k\}$ -reconstruction, or the so-called *reconstruction from below*, was introduced in 1970 by Fraïssé [18]. On the other hand, Pouzet [44] introduced the $\{-k\}$ -reconstruction in 1978, or the so-called *reconstruction from above*. These two types of reconstruction are closely related as was already established in 1976 by Pouzet [43, p. 131 case IV-1.2.1] with the following:

Lemma 1 (Pouzet [43, p. 131 case IV-1.2.1]). *Let R and R' be two relations on the same ground set E . If R and R' are $\{q\}$ -hypomorphic, where $1 \leq q \leq |E| - 1$, then for $p = 1, \dots, \min(q, |E| - q)$, R and R' are $\{p\}$ -hypomorphic.*

1.5. Ulam-reconstruction versus Fraïssé-reconstruction

Ulam-type reconstruction can be stated as follows: two relations R and R' have the same family—up to isomorphism—of sub-relations on k elements, for every k belonging to an integer set I . Does this imply that R and R' are isomorphic?

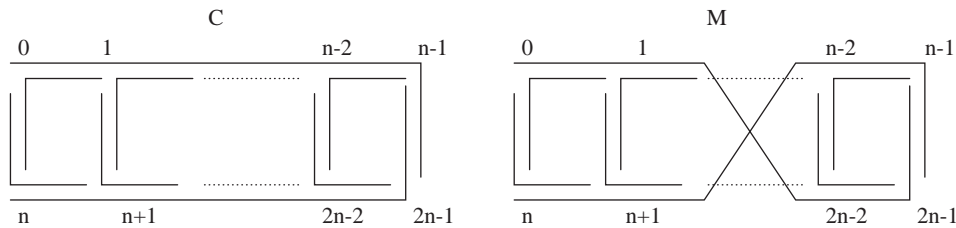


Fig. 1. The ternary relations C and M are not isomorphic, and for every strict subset X of $\{0, 1, \dots, 2n-1\}$ we have that $C[X] \simeq M[X]$.

Fraïssé-type reconstruction can be stated as follows: two relations R and R' on the same ground set E have, for any subset of E on k elements, the same sub-relations—up to isomorphism—, for every k belonging to an integer set I . Does this imply that R and R' are isomorphic?

If the two approaches are fundamentally different for arbitrary families of sub-relations, that is for arbitrary integer sets I , when $I = \{|E| - 1\}$, the $\{|E| - 1\}$ -reconstruction and the Ulam-reconstruction are then identical. A glimpse of that difference can be illustrated with $I = \{2\}$. Indeed, graphs (irreflexive symmetric binary relations) are clearly $\{2\}$ -reconstructible (in Fraïssé's sense), while any two graphs with the same number of edges will have the same family—up to isomorphism—of sub-graphs on 2 elements, so Ulam-type reconstruction does not hold for $k = 2$.

1.6. Negative results

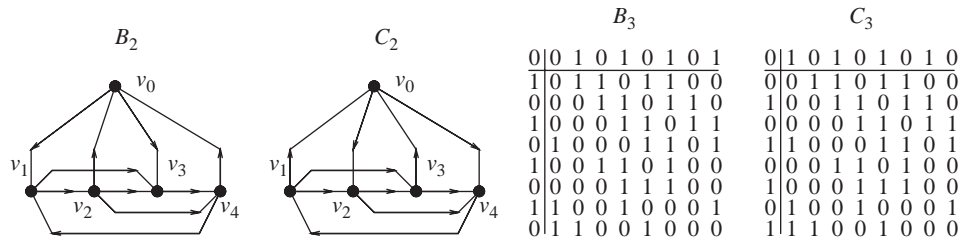
1.6.1. Ternary relations

The reconstruction of k -ary relations admits, for any $k \geq 3$, a negative answer given by Pouzet in 1979 with an infinite family of pairs of counterexamples. Using the band of Moebius, Pouzet first shows that 3-ary relations are not reconstructible. Then, he extends this result to arbitrary k -ary relations with $k \geq 3$. Indeed, let R and S be two k -ary relations on a set E . For any $l \geq k$, let T and U be two l -ary relations on E defined by: for every element (x_1, \dots, x_l) of E^l , let $T(x_1, \dots, x_k, x_{k+1}, \dots, x_l) = R(x_1, \dots, x_k)$ and let $U(x_1, \dots, x_k, x_{k+1}, \dots, x_l) = S(x_1, \dots, x_k)$. Then, R and S are isomorphic if and only if T and U are isomorphic.

Theorem 2 (Pouzet [45]). *For $n \geq 2$, let $E = \{0, 1, \dots, 2n-1\}$. Let C and M be two ternary relations on E , such that: (i) $C(i, j, k) = +$ for $j = i + n \bmod 2n$ and $k = j + 1 \bmod n$ for $j \leq n-1$ and $k = (j + 1 \bmod n) + n$ for $n \leq j$; (ii) $M(i, j, k) = +$ for $j = i + n \bmod 2n$ and $k = j + 1 \bmod 2n$. Then, ternary relations C and M are not isomorphic, and for every strict subset X of E , the sub-relation of C induced by X , and the sub-relation of M induced by X , are isomorphic (Fig. 1).*

1.6.2. Binary relations

The reconstruction of binary relations admits a negative answer given, with an infinite family of pairs of indecomposable tournaments, by Stockmeyer in 1977. For each positive integer n , the tournament A_n with 2^n elements $V(A_n) = \{v_1, \dots, v_{2^n}\}$ is defined by $v_i v_j \in$

Fig. 2. Stockmeyer's counterexamples for $n = 2$ and 3.

$E(A_n)$ if and only if $\text{odd}(j - i) \equiv 1 \pmod{4}$, for $i \neq j$. For any nonzero integer k , $\text{odd}(k)$ is the quotient k divided by $2^{\text{pow}(k)}$, where $\text{pow}(k)$ is the largest integer i such that 2^i divides k . That is, for example, we have $\text{pow}(-1) = 0$ and thus $\text{odd}(-1) = -1$, and we have $\text{pow}(12) = 2$ and thus $\text{odd}(12) = 3$. From that definition, Stockmeyer exhibits, among others, the following properties:

Property 3 (Stockmeyer [55]).

- (i) A_n has a unique automorphism.
- (ii) For each integer $1 \leq k \leq 2^n$ the sub-tournaments $A_n - v_k$ and $A_n - v_{2^n - k + 1}$ are isomorphic.

Two new tournaments on $2^n + 1$ elements are then obtained from A_n :

- the tournament B_n is obtained by adding a new element v_0 dominating all elements with even indices and dominated by all elements with odd indices,
- the tournament C_n is obtained by adding a new element v_0 dominating all elements with odd indices and dominated by all elements with even indices.

The validity of the counterexamples follows then, from the fact that Property 3 (i) implies that B_n and C_n are not isomorphic, and from the fact that Property 3 (ii) implies that, for each integer $1 \leq k \leq 2^n$, the sub-tournaments $B_n - v_k$ and $C_n - v_{2^n - k + 1}$ are isomorphic.

Theorem 4 (Stockmeyer [55]). *Binary relations are not Ulam reconstructible* (Fig. 2).

1.7. Positive results

A lot of positive results for sub-classes of binary relations, and for sub-classes of symmetric binary relations, have been obtained by now. For the reader interested in them, some survey papers are indicated in Section 1.8.

1.7.1. Reconstruction from below, from above

Looking at the reconstruction from below of binary relations, Lopez obtains a positive answer, in 1978, by assuming the hypomorphy for all the small sub-relations on up to six elements. This lower bound is the best possible since in 1990, Fraïssé and Lopez [19] give

infinite families of pairs of binary relations being both non-isomorphic and $\{2, \dots, 5\}$ -hypomorphic.

Theorem 5 (Lopez [37]). *Reflexive binary relations, on $n \geq 6$ elements, are $\{2, \dots, 6\}$ -reconstructible.*

Looking to the reconstruction from above of binary relations, Lopez and Rauzy showed, in 1992, that they are $\{-4\}$ -reconstructible, as a consequence of the following results.

Theorem 6 (Lopez and Rauzy [38,39]). *Reflexive binary relations, on $n \geq 7$ elements, are $\{2, 3, 4, n-1\}$ -reconstructible.*

Looking at the reconstruction from below of order relations, Hagendorf obtains a positive answer, in 1992, by assuming the hypomorphy for all the small sub-relations on up to three elements. Note that this result can be obtained as a corollary of the $\{-1, 2\}$ -reconstruction of orders shown by Ille and Rampon in 1998. See Section 8.1 for details.

Theorem 7 (Hagendorf [22]). *Order relations, on $n \geq 4$ elements, are $\{2, 3\}$ -reconstructible.*

1.7.2. Decomposable binary relations

A binary relation with base E is said to be decomposable if there exists X a nontrivial subset of E (that is $1 < |X| < |E|$) such that $\forall x, y \in X, \forall z \in E - X$ we have $xz \in E \iff yz \in E$ and $zx \in E \iff zy \in E$. A class \mathcal{C} of relations is said to be recognizable if every relation $\{-1\}$ -hypomorphic to any of the elements of \mathcal{C} belongs to \mathcal{C} . In 1993 Ille [27], interested in the recognition of decomposable binary relations, obtains:

Theorem 8 (Ille [27]). *Decomposable binary relations on at least 12 elements are Ulam recognizable.*

1.8. Open questions

1.8.1. Tournaments

If the general case of tournaments has been negatively answered by Stockmeyer (see Section 1.6.2) some particular cases encounter a positive outcome. Among the recent ones the $\{-1\}$ -reconstruction of the $\{-1\}$ -monomorphic tournaments has been shown by El-Issawi [15] in 1996, and the reconstruction of diamondless tournaments on at least seven elements has been shown by Gnanvo and Ille [21] in 1988. One of the main remaining questions is the $\{-1\}$ -reconstruction of decomposable tournaments.

1.8.2. Graphs (symmetric binary relations)

Although the original question of Ulam is still not settled, some particular classes have been shown to be reconstructible and others have been shown to be recognizable. A vast literature has been produced on this topic and some very good complementary surveys are the following: Bondy [5] from 1991, Capobianco and Molluzzo [7] from 1978, Nash-Williams [41] from 1978, Bondy and Hemminger [6] from 1977.

1.8.3. Partially ordered sets

In 1985, Sands [48] recounts the origins of the interest in the reconstruction of partially ordered sets.

... A special case which is still open, and on which virtually nothing has been published, is the subject of this note: poset reconstruction.

Is every finite partially ordered set P of more than three elements determined by its collection $\{P - \{x\} : x \in P\}$ of (unlabelled) one-point deleted “subsets”?

... Although the problem must have occurred to many people, explicit references to it in the literature are exceedingly rare. It was posed by Davey at the 1981 Banff meeting on ordered sets [13, p. 859]. His (and my) knowledge of the problem goes back to the mid-1970s at the University of Manitoba, and it had been proposed by Rival and myself at the 1976 Oberwolfach meeting on universal algebra. The problem has independently been considered by Das, who has found the only nontrivial results on it to date. In his thesis [12] he proved that posets of more than three elements whose HASSE diagrams are trees reconstructible. The proof, unfortunately, has not been published and is apparently very long. He also found some reconstructible properties; for instance he proved that the number of elements of height i in a poset P can be determined from the $P - \{x\}$'s (see Proposition 1 of [11] for a statement of his result). He has published two papers [10,11] related to that problem, ...

... Another paper relevant to poset reconstruction is by Hyyrö ([25]; see also Bennet Manvel's review #5234 in MR 41 (1971)), in which it is shown that certain kinds of bipartite posets are reconstructible.

In the same note, Sands also addresses two problems (and particularly the problem of the max-reconstruction) which have been negatively answered in 1994 by Kratsch and Rampon [31] (see Section 8.2):

“Is every finite distributive lattice D determined by its collection of maximal ideals, i.e., by the collection $\{[0, x] : x \text{ is a coatom of } D\}$? This is equivalent to: Is every finite poset P determined by the collection $\{P - \{x\} : x \text{ is a maximal element of } P\}$?”

“Is every finite distributive lattice D determined by its collection of prime ideals, i.e., by the collection $\{[0, x] : x \text{ is a meet-irreducible of } D\}$? Here the order-theoretic version is: Is every finite poset P determined by the collection $\{P - [x) : x \in P\}$, where $[x) = \{y \in P : x \leq_P y\}$?”

2. Ordered set definitions

In this paper we use standard notions like in the books of Fishburn [16], of Trotter [56], or of Schröder [50]. The book of Fishburn has for main topics interval orders and interval graphs. The monograph of Trotter focuses on the dimension theory of ordered sets. In the

textbook of Schröder the Ulam-reconstruction and the fixed point property are often used to illustrate the notions examined in the different chapters.

A *partially ordered set* (or *order* for short) P is an ordered pair $(V(P), E(P))$, where $V(P)$ is a finite set, called the set of *elements* (or *vertices*) of P , and $E(P)$ is an irreflexive and transitive binary relation on $V(P)$, called the *relation* (or set of *edges* of) P . For $x, y \in V(P)$, $x <_P y$ (respectively $x \parallel_P y$) is used in the place of $(x, y) \in E(P)$ (respectively $(x, y), (y, x) \notin E(P)$) and $x \leq_P y$ signifies either $x <_P y$ or $x = y$. Following these notations, given two orders P and Q , a one-to-one correspondence $f : V(P) \rightarrow V(Q)$ is an isomorphism from P onto Q provided that for all $x, y \in V(P)$, $x <_P y$ if and only if $f(x) <_Q f(y)$.

With each subset X of $V(P)$ is associated the *sub-order* $(X, E(P) \cap (X \times X))$ of P induced by X , denoted by $P[X]$. For convenience, if $X \subseteq V(P)$ (respectively $x \in V(P)$), then the sub-order $P[V(P) - X]$ (respectively $P[V(P) - \{x\}]$) is denoted by $P - X$ (respectively $P - x$).

The *dual* of P is the order $P^d = (V(P), E(P^d))$, where for $x, y \in V(P)$, $(x, y) \in E(P^d)$ if and only if $(y, x) \in E(P)$.

The *comparability graph* of P is the graph $G(P) = (V(P), E(G(P)))$, where for $x, y \in V(P)$, $\{x, y\} \in E(G(P))$ if and only if either $x <_P y$ or $y <_P x$. P is said to be *disconnected* (respectively, *connected*) if its comparability graph is a disconnected (respectively, connected) graph. The *incomparability graph* of P is the graph $\overline{G(P)} = (V(P), E(\overline{G(P)}))$, where for $x, y \in V(P)$, $\{x, y\} \in E(\overline{G(P)})$ if and only if $x \parallel_P y$. A subset X of $V(P)$ is called an *anti-chain* (respectively a *chain*) if elements of X are pairwise incomparable (respectively comparable).

The *cover relation* of P is denoted by \prec_P , that is for $x, y \in V(P)$ we have $x \prec_P y$ if and only if $x <_P y$ and $\nexists z \in V(P)$ such that $x <_P z <_P y$. The *transitive reduction* of P is the directed graph of its cover relation $Tr(P) = (V(P), E_{Tr}(P))$ (i.e. $(x, y) \in E_{Tr}(P)$ whenever $x \prec_P y$). A *covering sub-order* of P is any induced sub-digraph of its transitive reduction. The *covering graph* of P is the graph $C(P) = (V(P), E_C(P))$ where $\{x, y\} \in E_C(P)$ if and only if either $x \prec_P y$ or $y \prec_P x$.

An element x of $V(P)$ is *minimal* (respectively *maximal*) provided that for all $y \in V(P)$, $y \leq_P x$ (respectively $x \leq_P y$) implies that $x = y$. We denote by $\text{Min}(P)$ (respectively $\text{Max}(P)$) the set of minimal (respectively maximal) elements in P .

For $x \in V(P)$ we denote by $\downarrow_P^l x$, $\downarrow_P^1 x$ and $\downarrow_P^{\text{im}} x$ respectively the *predecessor set*, the *ideal* (or *closed predecessor set*), and the *lower cover set* of x in P . That is $\downarrow_P^l x = \{y \in V(P), y <_P x\}$, $\downarrow_P^1 x = \downarrow_P^l x \cup \{x\}$ and $\downarrow_P^{\text{im}} x = \{y \in V(P), y \prec_P x\}$. Similarly we use $\uparrow_P^l x$, $\uparrow_P^1 x$ and $\uparrow_P^{\text{im}} x$ for respectively the *successor set*, the *filter* (or *closed successor set*) and the *upper cover set* of x in P . For convenience, $|\uparrow_P^1 x|$ (respectively $|\downarrow_P^1 x|$) is denoted by $f_P(x)$ (respectively $i_P(x)$). These notations are extended to any subset X of $V(P)$ for the predecessor sets by $\downarrow_P^l X = (\bigcup_{x \in X} \downarrow_P^l x) - X$, $\downarrow_P^1 X = \bigcup_{x \in X} \downarrow_P^1 x = \downarrow_P^l X \cup X$ and $\downarrow_P^{\text{im}} X = (\bigcup_{x \in X} \downarrow_P^{\text{im}} x) - X$. The same extensions hold for the successor sets.

The *rank* of an element $x \in V(P)$, denoted $\text{rank}_P(x)$, is the length of a largest chain of P ending in x . Thus, if x is a minimal element of P then $\text{rank}_P(x) = 1$. Clearly, all elements of the same rank form an anti-chain of P and comparable elements have different rank. The i -th level of P , denoted by $L_i(P)$, is the subset of $V(P)$ containing all elements with rank

i in P : $L_i(P) = \{x \in V(P) : \text{rank}_P(x) = i\}$. The *height* of P , denoted $h(P)$, is one less than the maximal cardinality of a chain in P . The *width* of P , denoted $w(P)$, is the maximal cardinality of an anti-chain in P .

We continue this section with some of the notions and some of the properties concerning the decomposition of orders. Let P be an order. A subset X of $V(P)$ is an *interval* (or an autonomous subset or an homogeneous subset or a module) of P provided that for every $y \in V(P) - X$, either for all $x \in X$, $x <_P y$ or for all $x \in X$, $y <_P x$ or for all $x \in X$, $x \parallel_P y$. This definition is a generalization of the classic notion of the interval of total order. Notice also that an interval of an order P does not necessarily induce a connected subgraph of $G(P)$. Given an order P , \emptyset , $V(P)$ and $\{x\}$, where $x \in V(P)$, are intervals of P , which are called *trivial*. An order is then said to be *indecomposable* (or *prime*) provided that all of its intervals are trivial. Otherwise, it is said to be *decomposable*.

The following strengthening of the notion of interval, introduced by Gallai in [20], is also useful. Let P be an order. A subset X of $V(P)$ is a *strong interval* of P provided that X is an interval of P satisfying: for any interval Y of P , if $X \cap Y \neq \emptyset$, then $X \subseteq Y$ or $Y \subseteq X$. In all that follows, $\mathcal{S}(P)$ denotes the family of strong intervals X of P fulfilling: $X \neq V(P)$ and for every strong interval Y of P if $X \subseteq Y$, then $Y = X$ or $Y = V(P)$. We next introduce the notions of quotient and of lexicographical sum. Given an order P , a partition S of $V(P)$, all of the elements of which are intervals of P , is called an *interval partition* of P . For such a partition, define the *quotient* $P/S = (S, E(P/S))$ of P by S as follows: for $X \neq Y \in S$, $(X, Y) \in E(P/S)$ if and only if for $x \in X$ and for $y \in Y$, $(x, y) \in E(P)$. The inverse operation of the quotient is the *lexicographical sum* defined as: let P be an order, with any $x \in V(P)$ is associated an order P_x so that for $x \neq y \in V(P)$, $V(P_x) \cap V(P_y) = \emptyset$. The lexicographical sum of the P_x 's over P is the order $P(P_x; x \in V(P)) = \left(\bigcup_{x \in V(P)} V(P_x), E(P(P_x; x \in V(P))) \right)$ defined in the following manner: given $a \neq b \in \bigcup_{x \in V(P)} V(P_x)$, $(a, b) \in E(P(P_x; x \in V(P)))$ provided that either $x = y$ and $(a, b) \in E(P_x)$ or $x \neq y$ and $(x, y) \in E(P)$, where x and y are the vertices of P such that $a \in V(P_x)$ and $b \in V(P_y)$.

The following results of Gallai [20] allow for the description of the decomposition of orders.

Theorem 9 (Gallai [20]). *Let P be an order. One of the following conditions is satisfied:*

- (i) $G(P)$ is disconnected, $\mathcal{S}(P)$ is the interval partition of P consisting of the connected components of $G(P)$ and $P/\mathcal{S}(P)$ is empty.
- (ii) $\overline{G(P)}$ is disconnected, $\mathcal{S}(P)$ is the interval partition of P consisting of the connected components of $\overline{G(P)}$ and $P/\mathcal{S}(P)$ is a total order.
- (iii) $G(P)$ and $\overline{G(P)}$ are connected, $|\mathcal{S}(P)| \geq 4$ and $P/\mathcal{S}(P)$ is indecomposable.

Theorem 10 (Gallai [20]). *Given orders P and P' , if $G(P) = G(P')$ and if P is indecomposable, then $P' = P$ or $P' = P^d$.*

Proposition 11 (Gallai [20]). *Given orders P and P' , if $G(P) = G(P')$, then P and P' have the same strong intervals and, consequently, $\mathcal{S}(P) = \mathcal{S}(P')$.*

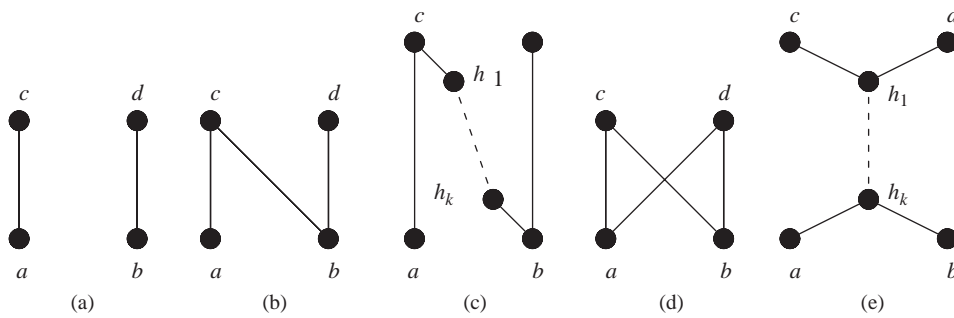


Fig. 3. (a) The $2 \oplus 2$ order; (b) the N order; (c) the N_k order; (d) the 4-crown order; (e) the X_k order. Note that the N order is actually the N_0 order, and that the 4-crown order is indeed the X_0 order.

3. Fundamental notions

For the Ulam reconstruction of orders, instead of using the hypomorphic approach, we mainly adopt the classical notions of deck, cards, \dots , which are used in the Ulam graph reconstruction. We present, in the following, their formal definitions within the formalism of the order theory (Fig. 3).

- (i) The *deck* of an order P is the family, $(P - x)_{x \in V(P)}$, of its unlabelled one-element-deleted sub-orders. Any unlabelled sub-order $P - x$ is said to be a *card* of P .
- (ii) An order Q is a *reconstruction* of an order P if there exists a bijection $\sigma : V(P) \rightarrow V(Q)$ such that for every $x \in V(P)$ holds: $P - x \simeq Q - \sigma(x)$.
- (iii) A parameter (or a function) defined on all orders is said to be *reconstructible* if it has the same value for all the reconstructions of any order.
- (iv) An order is said to be *reconstructible* if it is isomorphic to all of its reconstructions.
- (v) A class \mathcal{C} of orders is said to be *recognizable* if all the reconstructions of its elements belong to \mathcal{C} . Note that cards might not belong to \mathcal{C} (if the class \mathcal{C} is not hereditary).
- (vi) A class \mathcal{C} of orders is said to be *weakly reconstructible*, if for every order P in \mathcal{C} all the reconstructions of P , belonging to \mathcal{C} , are isomorphic to P .
- (vii) A class of orders is said to be *reconstructible* if all its elements are reconstructible.

Remark 12. A class \mathcal{C} of orders, which is recognizable and weakly reconstructible, is then reconstructible. This leads to a classical and powerful two-step procedure for proving that order classes are reconstructible.

Given the deck of an order P , that is, given the family $(P_x)_{x \in V(P)}$, we can construct a family $(R_x)_{x \in V(P)}$ of binary relations, where each R_x is isomorphic to a binary relation on $V(P) - x$. When the construction of such a family is appropriate, we obtain the deck of the binary relation associated to P . Such transformations of the original deck allow some

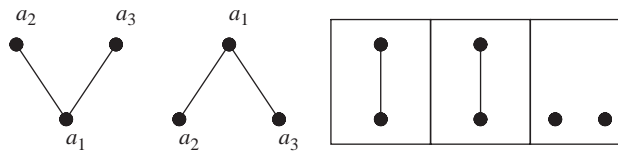


Fig. 4. The \vee -order and the \wedge -order, two non-isomorphic orders with the same deck.

interesting simplifications in proving reconstruction results. We present here three natural and useful such transformations.

- (i) The *dual deck* $(P_x^d)_{x \in V(P)}$.
- (ii) The *comparability graph deck* $(G(P_x))_{x \in V(P)}$.
- (iii) The *incomparability graph deck* $(\overline{G(P_x)})_{x \in V(P)}$.

It is a matter of routine to check that P has deck $(P_x)_{x \in V(P)}$ if and only if P^d has deck $(P_i^d)_{i \in [n]}$. Thus, an order is reconstructible if and only if its dual order is reconstructible. Furthermore, if a parameter is reconstructible, and if the parameter on the dual of an order is of interest, then the *dual parameter* is also reconstructible. The number of minimal elements and the number of maximal elements of an order are an important example of dual parameters. Also, one can easily check that, if P has deck $(P_x)_{x \in V(P)}$ this implies both that $G(P)$ has deck $(G(P_x))_{x \in V(P)}$, and that $\overline{G(P)}$ has deck $(\overline{G(P_x)})_{x \in V(P)}$. This allows us to use, for the reconstruction of orders, the fact that certain classes of graphs and certain graph parameters are reconstructible.

Remark 13. Due to the pair of orders given in Fig. 4, the question of the reconstruction of orders is open for orders on at least four elements. Implicitly, in the remainder of the paper, all orders that we consider for the Ulam reconstruction have now at least four elements.

4. Infinite orders

As noticed by Ille and Rampon in [29], the fact that infinite orders are not reconstructible is a direct consequence of the counterexample to the reconstruction of infinite graphs given in 1972 by Nešetřil [42] and independently by Fisher et al. [17]. This counterexample consists of a pair of graphs: one being a connected tree where each element has a countable degree, and the other being two copies of this tree. Note that any tree can be transformed into the transitive reduction of a height one order, simply by orienting all its edges from one color class to the other. Consequently, height one orders are not reconstructible.

In the case of infinite binary relations, the reconstruction question has an interesting extension, due to Halin, and which has been recorded in 1977 by Bondy and Hemminger [6]. Halin's observation was that all the known non-reconstructible pairs of infinite graphs have the property that each graph is isomorphic to an induced subgraph of the other.

He then conjectured that if two graphs have the same deck, then each of them is an induced subgraph of the other. In the case of orders, as for graphs, this conjecture is still unsettled.

Conjecture 14 (Halin). *If two infinite orders have the same deck then each of them is a sub-order of the other.*

5. Reconstruction of parameters

Recall that a parameter is said to be reconstructible if it has the same value for all the reconstructions of any order.

5.1. Kelly's lemma

In 1957, Kelly introduced a very powerful combinatorial counting lemma, today known as Kelly's Lemma. This lemma allows one, for any graph, to reconstruct the number of its strict subgraphs isomorphic to a given graph. The original proof also holds for arbitrary binary relations. We give here the version of this lemma for arbitrary binary relations. This is particularly interesting when dealing with orders, since it allows us to use the same lemma for its sub-orders or for sub-graphs of its comparability graph.

Lemma 15 (Kelly [30]). *The number $s(S, R)$ of sub-relations of R isomorphic to S is reconstructible for any two binary relations with $|V(S)| < |V(R)|$. Furthermore, the number of sub-relations of R , which are isomorphic to S and contain a given element x of $V(R)$, is reconstructible.*

Proof (Hint). Each sub-relation of R isomorphic to S occurs in exactly $|V(R)| - |V(S)|$ one-element-deleted sub-relations of R . Thus we get $s(S, R) \cdot (|V(R)| - |V(S)|) = \sum_{x \in V(R)} s(S, R - x)$. Since the second part of the equality is clearly computable from the deck, the result follows. The number of sub-relations of R , containing the element x and that are isomorphic to S , is simply $s(S, R) - s(S, R - x)$. \square

As a first consequence Kelly's lemma allows us to establish that the class of order relations is recognizable among all binary relations.

Theorem 16. *Orders are recognizable.*

Proof. Let P be an order and let Q be any of its reconstructions. If Q is not an order, then either Q is not transitive, or Q contains a (directed) cycle. By Kelly's lemma applied to the 3-element subsets of P we obtain that Q is a transitive relation. By Kelly's lemma applied to all the strict subsets of P we deduce that if Q contains a cycle then it is unique and Hamiltonian. This contradicts the fact that Q is transitive and has at least four elements. \square

As other direct consequences the following parameters appear to be reconstructible.

Proposition 17 (Folklore). *For an order P , the following parameters are reconstructible:*

- (i) *The number of edges of its comparability graph $|E(G(P))|$.*
- (ii) *The degree of x in its comparability graph $G(P)$.*
- (iii) *The degree sequence of its comparability graph.*
- (iv) *Its height $h(P)$ and its width $w(P)$.*

Proof (Hint). Kelly's lemma directly gives $|E(G(P))|$, and consequently $h(P)$ and $w(P)$. The degree of x in $G(P)$ is obtained from the card $P - x$. \square

5.2. Ideal-sizes and filter-sizes

In 1973, using a quite involved proof, Manvel shows that the degree pair sequence of a digraph is reconstructible. Recall that the degree pair sequence of a digraph $G = (V(G), E(G))$ is the family $(d^+(x), d^-(x))_{x \in V(G)}$ where $d^+(x)$, respectively $d^-(x)$, is the out-degree, respectively the in-degree, of x in G . Applied to orders, the in-degree corresponds to ideal size, and the out-degree corresponds to filter size. Since filters and ideals allow us to express a lot of the structure of an order (even all of the structure, for example with the ideal lattice), the reconstruction of such a parameter is particularly interesting for orders. Consequently, as we discuss in Section 5.3, the result of Manvel has been strengthened when restricted to the class of orders.

Theorem 18 (Manvel [40]). *The degree pair sequence of a digraph with at least 5 elements is reconstructible.*

Given an order P , for any $x \in V(P)$, we have that $|\uparrow_P^1 x| = d^+(x) + 1$ and $|\downarrow_P^1 x| = d^-(x) + 1$. Thus, as announced previously, just by rewriting Manvel's result in terms of order parameters, and by selecting particular sub-sequences, we obtain the reconstruction of some interesting parameters.

Corollary 19. *For any order, the following parameters are reconstructible:*

- (i) *Its (ideal, filter)-size pair sequence.*
- (ii) *The number of its maximal elements and the number of its minimal elements.*
- (iii) *The ideal-size sequence of its maximal elements.*
- (iv) *The filter-size sequence of its minimal elements.*

5.3. Ideals and filters

Recently, in 2000, Schröder extends, for the sub-class of orders, the result of Manvel we present in Section 5.2. He shows that the number of neighborhood-orders (sub-orders induced by a neighborhood) isomorphic to a given order is reconstructible. That is, he proves, for an order P , the reconstruction of the neighborhood deck, or equivalently, of the family $(P[\downarrow_P^1 x \cup \uparrow_P^1 x])_{x \in V(P)}$ of its unlabelled neighborhood-sub-orders. This result has been already announced by Das in 1976 under the assumption of the set-reconstruction.

See Section 8.5 for more details on this type of reconstruction. Thus, Das's result implies Schröder's result. However, on the one hand Das's results, which he announced in a 1979 paper (see [10]) as a submitted paper from 1976, is still unpublished at this time. On the other hand, the proof given by Schröder is simple and made a clever use of Kelly's Lemma. Thus we choose to record here Schröder's result.

The reconstruction of such a parameter is interesting, for the reasons evoked in Section 5.2, and also since it allows us to deduce a lot of reconstructible parameters for orders. Most of these parameters were already known to be reconstructible, but with ad hoc proofs, and now a global proof is available.

The result of Schröder relies on the notion of generating element: an element x is said to be *generating* in an order P whenever $\downarrow_P x \cup \uparrow_P x = V(P)$. Thus, a sub-order Q of P is a neighborhood-order of P under the two following conditions: (i) Q has a generating element, say x , and (ii) Q is not a strict sub-order of any sub-order of P having also x for generating element.

Theorem 20 (Schröder [49]). *The neighborhood deck of an order is reconstructible.*

Proof (*Hint*). Denote by $[Z, y]$ an order Z having y for generating element. Given an order P , let $s([Z, y], P)$ be the number of neighborhood-orders of P , isomorphic to Z , where the generating element of the neighborhood-order is mapped to y by the isomorphism. Then $s([Z, y], P) = n_a - n_b$, where n_a is the number of sub-orders of P , isomorphic to Z , having their generating element mapped to y , and where n_b is the sum, over all neighborhood-orders $[Q, x]$ of P with $|V(Q)| > |V(Z)|$, of the number of sub-orders of Q , containing x and isomorphic to Z , where x is mapped to y . As orders with a greatest element can be shown to be reconstructible without using the neighborhood deck (see Section 6.1), the result then follows from the fact that the maximal sized neighborhood-orders of P are directly given by Kelly's lemma. Indeed, for all sub-orders of P , say Z , having a generating element and being of maximal size with that property, we have that $s([Z, y], P) = s(Z, P)$ for every generating element y of Z . \square

By considering particular subsequences, Schröder deduced that the number of ideal-orders (i.e., sub-orders induced by a principal ideal) and the number of ideal-orders of the maximal elements isomorphic to a given order, are reconstructible. We summarize all that together with other reconstructible families of sub-orders and parameters, which are now also obviously reconstructible, in the following.

Corollary 21. *For any order, the following parameters are reconstructible:*

- (i) *Its ideal-orders sequence and its filter-orders sequence.*
- (ii) *The ideal-orders (respectively filter-orders) sequence of its maximal (respectively minimal) elements.*
- (iii) *The neighborhood-orders sequence of any fixed level of its rank decomposition.*
- (iv) *The ideal-size (respectively filter) sequence of any fixed level of its rank decomposition.*
- (v) *Its level-structure, that is the number of elements in any fixed level of its rank decomposition.*

5.4. Maximal cards

In 1994, Kratsch and Rampon introduce the notion of maximal card: a card of the deck of P is said to be *maximal* (respectively, *minimal*) if this card is isomorphic to a sub-order $Q - x$ with $x \in \text{Max}(Q)$ (respectively, $x \in \text{Min}(Q)$) for any reconstruction Q of P . Such cards are interesting to establish the reconstruction of parameters (see for example Section 5.5), and of order classes (see for example Section 6.4). They have shown that for every order, one maximal (respectively, minimal) card can be determined from its deck. They have also shown that for this card, the maximal (respectively minimal) elements, that are not maximal elements in every reconstruction, can also be determined. In fact, as it appears in their proof, a particular maximal (respectively minimal) card can always be determined. This card is such that the deleted element is of minimal ideal (respectively filter) size, over all the maximal (respectively minimal) elements of every reconstruction. We summarize all of this in the next lemma. But first, we have to state one more definition that they introduce to obtain these results.

A *hair* of length $l \geq 0$ in an order P is the sub-order of P induced by the set $\{u_0, u_1, u_2, \dots, u_l\}$, such that:

- (i) $u_0 \in \text{Max}(P)$
- (ii) For every $i \in \{1, 2, \dots, l\}$, u_i is the only immediate predecessor of u_{i-1} in P .
- (iii) For every $i \in \{1, 2, \dots, l-1\}$, u_{i-1} is the only immediate successor of u_i .
- (iv) u_l has either not exactly one immediate predecessor or at least two immediate successors.

Lemma 22 (Kratsch and Rampon [32]). *Let P be an order and let Q be any reconstruction of P . Then from the deck one can determine a maximal card $P^* = Q - x$ such that:*

- (i) $x \in \text{Max}(Q)$ and $i_Q(x) = \min\{i_Q(w) : w \in \text{Max}(Q)\}$.
- (ii) The maximal elements of the card P^* , that are not maximal elements in Q , can also be determined.

Proof (Hint). Recall that $|\text{Max}(P)|$ and $i_1 \geq \dots \geq i_{|\text{Max}(P)|}$, the decreasing ideal size sequence of the maximal elements of P , are reconstructible. We only have to consider the cards such that the $|\text{Max}(P)| - 1$ first entries, of the decreasing ideal size sequence of their maximal elements, correspond to those of P . Consider the set of such cards, such that, either the number of their maximal elements is different from $|\text{Max}(P)|$, or the last entry, in the decreasing ideal size sequence of their maximal elements, is different from $i_{|\text{Max}(P)|} - 1$. If this set is non-empty take any element in it. Otherwise, take among the previously selected cards, a card whose (unique) hair, having $i_{|\text{Max}(P)|} - 1$ for ideal size of its top element, is of minimal length. \square

Remarks 23.

- (i) The determination of such a maximal card P^* does not enable a reconstruction of P , since for $P^* = P - x$ with $x \in \text{Max}(P)$ we do not know the immediate predecessors of x , that are not maximal elements of $P - x$.

- (ii) The same holds for a minimal card $P^* = Q - x$, with $x \in \text{Min}(Q)$ and $f_Q(x) = \min\{f_Q(w) : w \in \text{Min}(Q)\}$.

The following question is strongly related to the max-reconstruction of orders. Even if we know that such reconstruction is not possible, progress in that direction would lead to a better understanding of what could be the key parameters for the order reconstruction. See Section 8.2 for more details on the max-reconstruction of orders.

Question 24 (Kratsch and Rampon [32]). *Can one determine more than one, or even all minimal (respectively, maximal) cards from the deck of any order?*

5.5. Covering sub-orders

Recall that a covering sub-order, of a given order, is any induced sub-graph of its transitive reduction. As such sub-orders are used to define order classes, and since they are representative of a given orientation of the comparability graph, it seems interesting to obtain a Kelly-type lemma for covering sub-orders. In 1994, Kratsch and Rampon establish that the number of covering edges, that is of covering sub-orders isomorphic to a two-element total order, is reconstructible. Note that this number is not a comparability invariant. That is, distinct orders, with the same comparability graph, have distinct numbers of covering edges. At the same time, they also ask for the reconstruction of arbitrary covering sub-orders. But, until now, no progress has been made in that direction. The proof that they have proposed uses the notion of minimal cards, and the notion of 2-transitivity edges. A 2-transitivity edge of an order is any of its non-covering edges which become a covering edge in one of its one-element-deleted sub-orders. The proof, that we outline here, is one that we find more direct and was told to us in 1999 by Schröder (see also Schröder [49] for another easy proof).

Theorem 25 (Kratsch and Rampon [32]). *The number of covering edges of an order is reconstructible.*

Proof (Hint). Consider a maximal card P^* as obtained in Lemma 22. Recall that, by Theorem 20, the number of order ideals of the maximal elements, isomorphic to a given order, is reconstructible. To conclude, notice that the number of immediate predecessors of a maximal element of an order is the number of its immediate predecessors in the sub-order induced by its ideal. \square

Question 26 (Kratsch and Rampon [32]). *For every order P and every order T with $|V(T)| < |V(P)|$ is the number of occurrences of T as covering sub-order the same for all reconstructions of P ?*

6. Some reconstructions

Recall that a class of orders is reconstructible if all its elements are reconstructible.

6.1. Orders with a least element

An elementary but fundamental structure associated to an order is its level-decomposition. This level-decomposition allows Kratsch and Rampon to obtain in [32] a simple proof for the reconstruction of orders with a least element. This result seems to be obtained for the first time by Das in 1973.

Theorem 27 (Das [8]). *Orders with a least or a greatest element are reconstructible.*

Proof (*Hint*). Due to duality, we consider only orders with a least element. To show they are recognizable, it is sufficient to count the number of cards with a least element. Let P be an order with a least element, and let k be the least level where one of its one-element deleted sub-orders has at least two elements. Note that any one-element deleted sub-order Q of P is isomorphic to the sub-order of P obtained by deleting its least element if and only if the first level of Q with at least two elements is k . This insures the reconstruction of such orders. \square

According to Birkhoff (see [3, p. 6]), recall that a *lattice* is an order L any two of whose elements have a greatest lower bound or “meet” denoted by $x \wedge y$, and a least upper bound or “join” denoted by $x \vee y$. Since finite lattices have a least element, we immediately obtain the following.

Corollary 28. *Lattices are reconstructible.*

However the reconstruction of lattices did not use the real intrinsic structure of lattices. Thus, a more natural question, regarding interest in the lattice structure, is the reconstruction of truncated lattices. Some steps towards a positive answer are given in [47] with the recognition of truncated lattices (see Section 6.6), with the reconstruction of truncated lattices with a 4-crown as sub-order, and with the reconstruction of semi-modular truncated lattices.

Remark 29. Implicitly, in the remainder of the paper all orders, that we consider for the Ulam reconstruction, have now at least two maximal elements and two minimal elements.

6.2. Disconnected orders

Recall that an order is said to be disconnected if its comparability graph is disconnected. Recall that an order is said to be co-disconnected if its incomparability graph (i.e., its co-comparability graph) is disconnected. In 1964, Harary [23] showed that disconnected graphs are reconstructible. As noticed by Kratsch and Rampon in [32], along the same lines one can also show that disconnected orders are reconstructible. The reconstruction of disconnected orders seems to be obtained for the first time by Das in 1973 with a different approach than the one of Harary. Note that in fact Harary’s proof also holds for any disconnected binary relation.

Theorem 30 (Harary [23], Das [8]). *Disconnected orders are reconstructible.*

Proof (Hint). To show they are recognizable, it is sufficient to count the number of connected cards. Let Q be a connected order of size k . Notice that the number of connected components of an order P , isomorphic to Q , is the difference between (i) the number of sub-orders of P isomorphic to Q , and (ii) the number of sub-orders, isomorphic to Q , in all the connected components of P on more than k elements. To show they are reconstructible, it then suffices to notice that the maximal sized connected components of P are directly given by Kelly's Lemma. \square

The reconstruction of co-disconnected orders seems to be shown for the first time by Kratsch and Rampon in 1994:

Theorem 31 (Kratsch and Rampon [32]). *Co-disconnected orders are reconstructible.*

Proof (Hint). They are recognizable since disconnected graphs are recognizable, and since the deck of an order gives the deck of its co-comparability graph. The uniqueness of the series decomposition (see Theorem 9 and Proposition 11) insures that co-disconnected orders are reconstructible. Recall that we only have to consider orders with at least two maximal and two minimal elements. Let P be such an order. Then, any card of P with the smallest number of elements, in the lower interval of its series composition, is obtained by the deletion of an element belonging to the lower interval of the series composition of every reconstruction of P . Consequently all the other intervals of the series composition of this card are for every reconstruction of P intervals of its series composition. This is also true for a card with the smallest number of elements in the upper interval of its series composition. From these two cards we can now deduce that every reconstruction of P is unique up to isomorphism. \square

As an easy consequence of these two previous theorems we obtain:

Corollary 32. *Series-parallel orders are reconstructible.*

Proof (Hint). Series-parallel orders are either disconnected or co-disconnected. \square

Remark 33. Implicitly, in the remainder of the paper all orders, that we consider for the Ulam reconstruction, are now neither disconnected nor co-disconnected.

6.3. Interval orders

An order $P = (V(P), <_P)$ is said to be an *interval order* if it can be represented by assigning a real interval $I_x = [l(x), r(x)]$ to each element $x \in V(P)$, such that $x <_P y$ if and only if $r(x) <_{\mathbb{R}} l(y)$ for all $x, y \in V(P)$. Using the notion of maximal card (see Section 5.4), in 1994, Kratsch and Rampon establish that interval orders are reconstructible. However, this result can be obtained in a much simpler way, directly from the results of Manvel (see Corollary 19 (i)), and the representation theorem for interval orders of Fishburn (see

Fishburn [16, Chapter 2] and particularly the proof of Theorem 6 on p. 29, see also Bogart [4] for a short proof).

Theorem 34 (Fishburn [16]). *Let P be an order, let $U(P) = \{\uparrow_P^l x : x \in V(P)\}$ and $D(P) = \{\downarrow_P^l x : x \in V(P)\}$. For every $x \in V(P)$, let $l(x) = |\{d \in D(P) : d \subset \downarrow_P^l x\}|$, and let $r(x) = |\{u \in U(P) : \uparrow_P^l x \subset u\}|$. The following conditions are then equivalent:*

- (i) P is an interval order.
- (ii) P has no sub-order isomorphic to the $2 \oplus 2$ order (see Fig. 3(a)).
- (iii) $D(P)$ is linearly ordered by inclusion.
- (iv) $U(P)$ is linearly ordered by inclusion.
- (v) $x <_P y \iff r(x) <_{\mathbb{N}} l(y)$.

Theorem 35 (Kratsch, Rampon [32]). *Interval orders are reconstructible.*

Proof. Interval orders are recognizable by Theorem 34 (ii) and Kelly's Lemma. Theorem 34 (iii) implies that for interval orders $\downarrow_P^l x \subset \downarrow_P^l y$ if and only if $|\downarrow_P^l x| <_{\mathbb{N}} |\downarrow_P^l y|$ and $\downarrow_P^l x = \downarrow_P^l y$ if and only if $|\downarrow_P^l x| = |\downarrow_P^l y|$. The same holds for successors sets with Theorem 34 (iv). Then, Corollary 19 (i) implies that the pair sequence $(l(x), r(x))_{x \in V(P)}$ is reconstructible. Then, Theorem 34 (v) allows us to conclude that all the reconstructions of a given interval order have one identical interval representation on the real line (the one given by $(l(x), r(x))_{x \in V(P)}$). Consequently all the reconstructions of a given interval order are isomorphic. \square

6.4. Width two orders

Recall that an order has width k , if its maximal sized anti-chain is of size k . Using notion of hair and of maximal cards (see Section 5.4). In 1996, Kratsch and Rampon establish the reconstruction of width two orders.

Theorem 36 (Kratsch and Rampon [33]). *Orders of width at most two are reconstructible.*

Proof (Hint). Width two orders are recognizable by Kelly's Lemma. Let P be a width two order having $i_1 \geq i_2$ for decreasing ideal size sequence of its maximal elements. For any reconstruction Q of P , and thus for P too, denote by x_1, x_2 its maximal elements, and assume that $|\downarrow_Q^1 x_1| = i_1$. Let P^* be a maximal card as determined in Lemma 22. Two cases have then to be considered; Case 1: $|\text{Max}(P^*)| = 2$ and Case 2: $|\text{Max}(P^*)| = 1$.

For Case 1, assume that $P^* \simeq P - x_2$ and thus we have that $\text{Max}(P^*) = \{x_1, y_2\}$. Then, every reconstruction of P is obtained by adding to P^* a new maximal element, either having for immediate predecessor y_2 whenever $|\downarrow_{P^*}^1 y_2| = |\downarrow_P^1 x_2| - 1$, or having for immediate predecessors y_2 and z , where $z \in V(P^*) - \downarrow_{P^*}^1 y_2$ and is such that $|\downarrow_{P^*}^1 z - \downarrow_{P^*}^1 y_2| = (|\downarrow_P^1 x_2| - 1) - |\downarrow_{P^*}^1 y_2|$.

For Case 2, we have that $P^* \simeq P - x_2$, and thus we can determine the number of immediate predecessors of x_1 in P . By a case study on this number of immediate predecessors, either

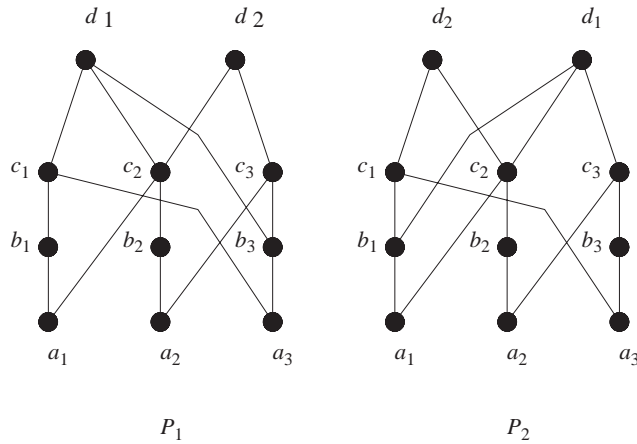


Fig. 5. Two non-isomorphic orders of width three such that $P_1 - x \simeq P_2 - x$ for any $x \in \{d_1, d_2\}$.

we can determine a maximal card, say P^o , isomorphic to $P - x_1$, or we can exhibit an argument which allows us to prove directly that all the reconstructions are isomorphic to P . Let y and z be the two maximal elements of P^o . The only non-direct case, for showing that P is reconstructible, is when both $|\downarrow_{P^o}^1 y| = |\downarrow_{P^o}^1 z| = |\downarrow_P^1 x_2|$ and $\text{rank}_{P^o}(y) = \text{rank}_{P^o}(z) = \text{rank}_P(x_2)$. Now we use the card $P - x_2$. Note that the immediate predecessor of the greatest element of $P - x_2$, whose rank is distinct from $\text{rank}_P(x_2)$, is also an immediate predecessor of x_2 in P . Then, using the same arguments as those used in Case 1, we can show that P is reconstructible. \square

The approach followed to obtain this result only uses maximal cards, and is then not accurate for orders of width at least four. Indeed, Kratsch and Rampon have shown in 1994, using a pair of width four orders, that orders are not max-reconstructible. For more details on the max-reconstruction see Section 8.2, and see Section 8.2.2 for the counterexample. This seemed to leave the possibility that the case of width three orders may be doable under this approach. But, in a recent paper from 2002, about the reconstruction of sub-classes of width three orders, Schröder [53] gives a pair of width three orders that are not max-reconstructible. We report this example in Fig. 5: to see that P_1 and P_2 are not isomorphic, notice that any isomorphism φ from P_1 onto P_2 would be such that both $\varphi(d_1) = d_1$ and $\varphi(d_2) = d_2$. But this is not compatible with the fact that every automorphism of $P_1[\{x_i : i \in \{1, 2, 3\}, x \in \{a, b, c\}\}] = P_2[\{x_i : i \in \{1, 2, 3\}, x \in \{a, b, c\}\}]$ induce a circular permutation on the sequence (c_1, c_2, c_3) .

6.5. N -free orders

N -free orders (or $\mathcal{C}AC$ -orders; see Leclerc and Monjardet [35,36]) are those orders such that every maximal (for inclusion) chain intersects every maximal (for inclusion) anti-chain, or, equivalently, are those orders having no N as covering sub-order (see Fig. 3(b)). In 1994

Kratsch and Rampon have shown that N -free orders are recognizable (for another proof see also Schröder [49]), but they leave unresolved the reconstruction question. Recently in 2000, Schröder succeeded to show that N -free orders are reconstructible.

Theorem 37 (Kratsch and Rampon [32]). *N -free orders are recognizable.*

Proof (Hint). Note that an N -free order P has at most $|V(P)| - |\text{Max}(P) \cup \text{Min}(P)|$ cards which are not N -free, and that a non- N -free order Q has at most $|V(Q)| - 4$ cards which are N -free. Thus it is sufficient to consider an N -free order P with exactly two maximal and two minimal elements. Let Q be a reconstruction of P and assume that Q is not an N -free order. Then Q has a unique N as covering sub-order, and moreover, all the cards $Q - x$ have to be N -free whenever x belongs to this N . Note that Lemma 22 insures the existence of a minimal card and of a maximal card. As these cards are N -free, the N of Q contains at least one maximal element and one minimal element of Q . If Q has more than five elements, and if all the elements of its N are either maximal or minimal elements, it is easy to establish that Q contains in fact more than one N . We are left with the case where one element of the N is neither a maximal element nor a minimal element of Q . Let z be such an element. Consider the N with labels as in Fig. 3. If $z = a$, then z has a predecessor, say w , which is incomparable to b . A case study, on the immediate successors of w in $Q - z$, allows us to deduce that either the card $Q - z$ is not N -free, or that Q contains at least two N 's. If $z = b$, then z has a predecessor, say w , which is incomparable to a . Again a case study, on the immediate successors of w in $Q - z$, allows us to deduce that either the card $Q - z$ is not N -free, or that Q contains at least two N 's. The remaining cases $z = c$ and $z = d$ follow by duality. \square

Theorem 38 (Schröder [51]). *N -free orders are reconstructible.*

Proof (Hint). Consider the N_k orders with labels as in Fig. 3. The proof relies on the two following facts.

Fact 1: If P is an indecomposable N -free order, then it has an N_1 order, as covering sub-order, such that $\uparrow_P^l b \cap \downarrow_P^l c = \{h_1\}$. Indeed if P is an indecomposable N -free order it contains an N as sub-order, and thus an N_k order as covering sub-order. Let N_l be a covering sub-order of P with the following property Π : (b, h_l, \dots, h_1, c) is the longest chain from b to c in P . If $l \neq 1$, then a careful case analysis allows us to exhibit an N_t covering sub-order with $t < l$ and fulfilling property Π (the element c of the N_t is the element h_1 of the N_l). Thus, there exists an N_1 fulfilling property Π , which insures the correctness of Fact 1.

Fact 2: If P is an N -free order having a card which is not N -free, then P is reconstructible. Indeed, if $P - x$ is not N -free, let $(N_i)_{i \in I}$ be the family of all its N 's as covering sub-orders. Consider that every N_i has its labels as in Fig. 3 with an added index i . Then, for any reconstruction of P , the added element must have all the c_i 's for immediate successors and all the b_i 's for immediate predecessors. It is then easy to check that, for any N -free order Q having an N_1 covering sub-order, all the immediate successors and all the immediate predecessors of h_1 belong to an N covering sub-order of $Q - h_1$. This insures the correctness of Fact 2.

Now, let P be a non-series-parallel N -free order. That is, $P = Q(P_x; x \in V(Q))$ and Q is an indecomposable N -free order. Due to Facts 1 and 2 it remains to consider the case where, for all the N_1 covering sub-orders of Q , $|V(P_{h_1})| \geq 2$. Let k be the minimum cardinality of such $V(P_{h_1})$. As $k \geq 2$, it is a matter of routine to show that P is uniquely determined—up to isomorphism—from a card $Z = Q(Z_x; x \in V(Q))$ having an N_1 covering sub-order with $|V(Z_{h_1})| = k - 1$. \square

6.6. Some truncated lattices

Recall that a finite order T is called a *truncated lattice* if there exists a lattice L isomorphic to \widehat{T} , where \widehat{P} is the order obtained by adding to P both a least element \perp , and a greatest element \top . Recently, in 2000, Rampon and Schröder were interested in the reconstruction of truncated lattices. They show that they are recognizable (see Section 7.6) but the reconstruction question still remains open. However, they were able to establish that some sub-classes were reconstructible, and among them, we present next the two principal ones.

First, recall that given an order P , its Dedekind–MacNeille completion is the set $V(DM(P)) = \{X \subseteq V(P) : X = X^{+-}\}$ ordered by inclusion, where X^+ , respectively X^- , denotes the set of upper, respectively lower, bounds of X . That is, $X^+ = \{y \in V(P) : \forall x \in X, x \leq_P y\}$, and $X^- = \{y \in V(P) : \forall x \in X, y \leq_P x\}$. It is well known that this order, denoted $DM(P)$, is a lattice such that the mapping $\varphi : V(P) \rightarrow 2^{V(P)}$, with $\varphi(x) = \downarrow_P^1 x$, is an embedding from P into $DM(P)$. That is, φ is an isomorphism from P onto $DM(P)[\varphi(V(P))]$. We will use the following well-known property of the Dedekind–MacNeille completion.

Property 39 (cf. Birkhoff [3, Exercise 5, p. 128]). *Let P be an order, and let L be a lattice. If P embeds into L , then $DM(P)$ embeds into L .*

Theorem 40 (Rampon and Schröder [47]). *Truncated lattices having a 4-crown as sub-order are reconstructible.*

Proof (*Hint*). Note that if a truncated lattice contains a 4-crown as sub-order, then it contains an X_k order, with $k \geq 1$, as covering sub-order (see Fig. 3). By Theorem 56 (see Section 7.6) truncated lattices are recognizable. Let T be a truncated lattice containing a 4-crown. It is not difficult to show the two following facts:

Fact 1: the least k such that T contains an X_k order as covering sub-order is reconstructible;

Fact 2: for X_k a covering sub-order and k being of minimal value, if its labels are as in Fig. 3 then any element h_i , with $2 \leq i \leq k$, has h_{i-1} for unique immediate successor in T , and any element h_i , with $1 \leq i \leq k - 1$, has h_{i+1} for unique immediate predecessor in T .

Now either the least k such that T contains an X_k order as covering sub-order is greater than 1, and then Fact 2 allows the uniqueness—up to isomorphism—of the reconstruction from any card which contains an X_{k-1} order as covering sub-order; or T contains an X_1 order as covering sub-order, and the uniqueness of the Dedekind–MacNeille completion (see Property 39) allows the uniqueness—up to isomorphism—of the reconstruction from any card which is not a truncated lattice. \square

To show that truncated semi-modular lattices are reconstructible, we need the notion of max-dominating element: an element x is said to be *max-dominating* in an order P whenever $x \in \text{Max}(P)$ and $\downarrow_P^L x = V(P) - \text{Max}(P)$. We also need two partial results that we group together in the following theorem.

Theorem 41 (Rampon and Schröder [47]).

- (i) *Truncated semi-modular lattices of height one are reconstructible.*
- (ii) *Let T be a truncated upper semi-modular lattice of height at least two that does not contain any four-crown. Then, either T has a max-dominating element, or T is co-disconnected.*

Now we can establish:

Theorem 42 (Rampon and Schröder [47]). *Truncated semi-modular lattices are reconstructible.*

Proof (*Hint*). By duality it is sufficient to consider the reconstructibility of truncated upper semi-modular lattices. By Theorems 40 and 41(i) we only need to consider truncated lattices of height greater than one, and which do not contain any four-crown. By Theorem 41(ii), either T is co-disconnected and then reconstructible (see Section 6.2), or T has a max-dominating element. But, since all the maximal elements of a truncated semi-modular lattice have the same rank, it is then easy to show that T is reconstructible. \square

7. Some recognitions

Recall that a class of orders is recognizable, if all the reconstructions of its elements belong to it. By Kelly's lemma, classes defined by a finite family of sub-orders are easily recognizable, simply by assuming that the orders have a sufficient number of elements.

7.1. Comparability graphs

Comparability graphs were characterized by families of forbidden subgraphs in 1967 by Gallai [20]. They are characterized by a family of 10 forbidden subgraphs on 7 vertices each, and by the following 8 infinite families of forbidden sub-graphs (see also Trotter and Moore [57]): C_{2n+1} for $n \geq 2$, J_n for $n \geq 2$, J'_n for $n \geq 3$, J''_n for $n \geq 2$, \overline{B}_n for $n \geq 1$, \overline{C}_n for $n \geq 6$, \overline{L}_n for $n \geq 1$, \overline{L}'_n for $n \geq 1$. This characterization was used by von Rimscha in 1983 to show that comparability graphs are recognizable (Fig. 6).

Theorem 43 (Von Rimscha [59]). *Comparability graphs are recognizable.*

Proof (*Hint*). By Kelly's lemma, to show that comparability graphs are recognizable, we only have to consider that the input graph is actually one of the forbidden subgraphs. Given a graph G , it then remains to show that any reconstruction of G is isomorphic to G . As the way of proving such a thing is similar for all the forbidden subgraphs, we only outline the case

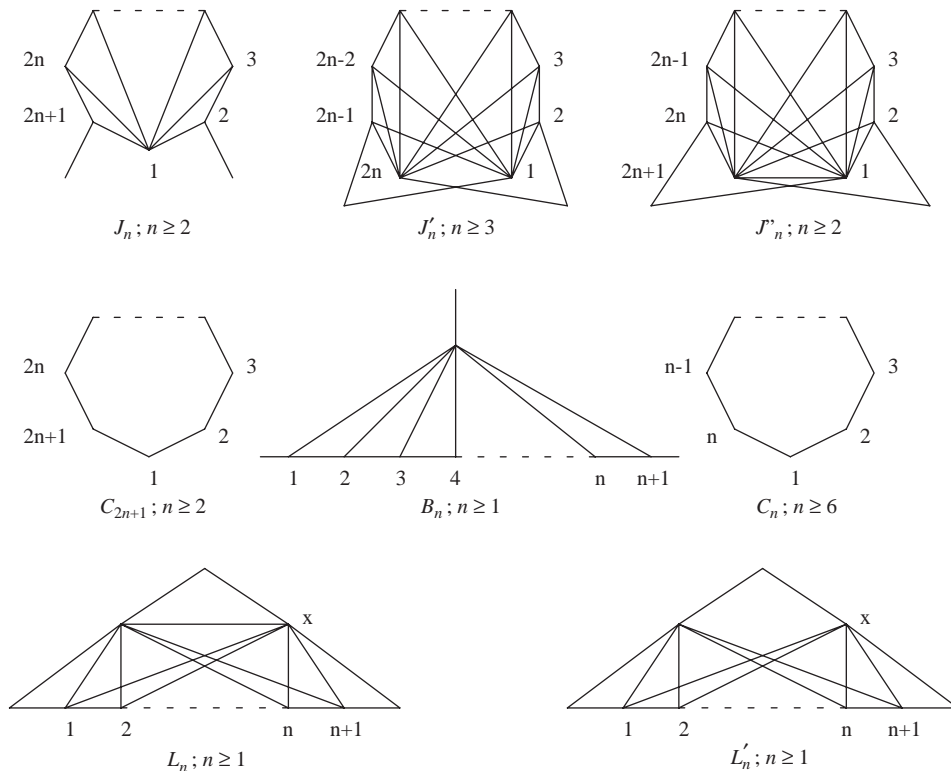


Fig. 6. The infinite families of forbidden subgraphs for comparability graphs. Recall that for the last four families the forbidden ones are $\overline{B_n}$, $\overline{C_n}$, $\overline{L_n}$ and $\overline{L'_n}$.

$G = \overline{B_n}$. Note that a graph is reconstructible if and only if its complement is reconstructible. As the deck of a graph gives the deck of its complement, we can assume that $G = B_n$. Recall that the degree sequence is reconstructible, and thus any reconstruction of G has for degree sequence $d_1 = d_2 = \dots = d_{n+1} = 3$, $d_{n+2} = d_{n+3} = d_{n+4} = 1$, and $d_{n+5} = n + 2$. Consider the unique (for n big enough) card in which the vertex of greatest degree has been deleted. Its degree sequence is then $d_1 = \dots = d_{n+1} = 2$, $d_{n+2} = d_{n+3} = 1$ and $d_{n+4} = 0$. From that card, there is only one possibility to obtain the degree sequence of G by adding a new vertex with degree $n + 2$. Consequently this leads to a unique—up to isomorphism—graph. \square

Recall that two-dimensional orders are those orders whose incomparability graph is also a comparability graph. As the deck of any order gives the deck of its incomparability graph, Von Rimscha's result implies that two-dimensional orders are recognizable. However, the reconstruction question is still open.

Question 44. *Are two-dimensional orders reconstructible?*

As suggested by one referee, since dimension is always at most width, notice that a positive answer to this previous question would imply Theorem 36.

7.2. Cycle-free orders

Triangulated graphs, that is graphs such that each cycle of length at least 4 has at least one chord, are characterized by the existence of a perfect vertex elimination scheme (see Dirac [14]). That is, given a graph G , G is triangulated if and only if there exists an ordering (x_1, \dots, x_n) of its vertices such that, for every i , the neighborhood of x_i in $G_i = G[\{x_i, \dots, x_n\}]$ is a clique. Using the fact that, by Kelly's lemma, the number of cliques of any size is reconstructible, in 1983 Von Rimscha established:

Theorem 45 (Von Rimscha [59]). *Triangulated graphs are recognizable.*

Proof (*Hint*). Let G be a triangulated graph, and let x be the first vertex in a perfect vertex elimination scheme of G . Then, x belongs to a clique of size k in G . By Kelly's lemma the number of cliques of size k is reconstructible. Then, in any reconstruction G' of G , the vertex x' such that $G' - x' \simeq G - x$ has for neighborhood a clique in G' . This implies that G' is triangulated. \square

Recall that cycle-free orders are those orders whose comparability graphs are triangulated. Again, as the deck of any order gives the deck of its comparability graph, Von Rimscha's result implies that cycle-free orders are recognizable. However, the reconstruction question is still open.

Question 46. *Are cycle-free orders reconstructible?*

7.3. Gallai's decomposition

One technique which is powerful for proving reconstruction results is to use induction. It is then important to investigate decompositions of orders based on partitions of their element sets. Among this kind of decomposition, the Gallai decomposition (see Section 2 for its formal definition and some of its properties) is particularly interesting.

Recall that Ille [27] shows that decomposable binary relations, on at least 12 elements, are recognizable, and thus that decomposable orders, on at least 12 elements, are recognizable. Another interesting result about Gallai's decomposition and reconstruction has been obtained in 1993 by Basso-Gerbelli and Ille. This result assumes that the orders under consideration have at least two non-trivial intervals. This condition could appear restrictive since one critical point, for the reconstruction of decomposable orders, is when they have a unique non-trivial interval of cardinality two. Nevertheless, it was a key point to establish that orders are $\{-1, 2\}$ -reconstructible (see Section 8.1).

Lemma 47 (Basso-Gerbelli, Ille [1]). *Let $P = Q(P_x; x \in V(Q))$ and $P' = Q'(P'_x; x \in V(Q'))$ be lexicographical sums fulfilling the following conditions:*

- (i) P and P' are $\{-1\}$ -hypomorphic.
- (ii) Q and Q' are indecomposable.
- (iii) $|\{x \in V(Q) : |V(P_x)| \geq 2\}| \geq 2$ and $|\{x \in V(Q') : |V(P'_x)| \geq 2\}| \geq 2$.

Then Q and Q' are isomorphic and for all $u \in \bigcup_{x \in V(Q)} V(P_x)$, $P_\alpha \simeq P'_\beta$, where $\alpha \in V(Q)$, $\beta \in V(Q')$ and $u \in V(P_\alpha) \cap V(P'_\beta)$.

As indicated at the beginning of this section the reconstruction of decomposable orders is still an open question. If its dual question on indecomposable orders does not seem to be easier, what about the following restrictions? A critical point, in an indecomposable order, is any point whose deletion gives a decomposable order. See Schmerl and Trotter [54], and Ille [26] for some studies of indecomposable orders and relations.

Question 48. *Are indecomposable orders, whose critical points are either minimal elements or maximal elements, reconstructible?*

One step towards a positive answer to the following question is given in Section 7.4.

Question 49. *Is there a short direct proof of the recognition of decomposable orders?*

7.4. Height one orders

One of the most natural and challenging classes of graphs to study is the one of bipartite graphs, but no progress has been obtained since Kelly [30] showed, in 1957, that trees are reconstructible. The difficulty of the bipartite graph reconstruction problem gives then interest to the study of some perhaps restricted version of this problem, like vertex-colored bipartite graphs or height one orders. In this former direction, Hyvärinen [25] studied, in 1968, with some success, the reconstruction of properly 2-vertex-colored bipartite graphs (see Section 8.2). In this latter direction, Rampon and Schröder [47] showed, in 2000, that truncated semi-modular lattices of height one are reconstructible (see Section 6.6).

Looking to Question 49 of Section 7.3, a little step towards a positive answer has been obtained in 1998 by Kratsch, Müller and Rampon.

Proposition 50 (Kratsch et al. [34]). *Decomposable height one orders on at least four elements are recognizable.*

Proof (Hint). Let P be a decomposable height one order, with $|V(P)| = n$, where P is both connected and co-connected. Note that, since P is decomposable then it has either n or $n - 2$ decomposable cards.

Case 1: P has n decomposable cards. Then all reconstructions of P are decomposable. Indeed, an indecomposable order Q , with $|V(Q)|$ decomposable cards, is either a height one interval order or a width two order (see Schmerl and Trotter [54] or Ille [26]). But, both interval orders and width two orders are reconstructible (see Kratsch and Rampon [32,33]).

Case 2: P has $n - 2$ decomposable cards. Then P has a unique non-trivial interval and this interval has two elements. Assume that this non-trivial interval is a subset of its minimal elements. Then either (a) the $n - 2$ decomposable cards have an interval, on two non-isolated minimal elements, or (b) only $n - 3$ decomposable cards have an interval, on two

non-isolated minimal elements. It is then easy to conclude by a case study, on the number of minimal elements of P . Indeed, any indecomposable order Q has at most $|V(Q)| - |\text{Min}(Q)|$ decomposable cards, with an interval on two non-isolated minimal elements. \square

In 1998 Kratsch et al. [34] noted that for any height one order having all its maximal, respectively minimal, elements with at least 2 predecessors, respectively successors, then all its maximal cards can be determined from its deck. This leads to the following question.

Question 51. *Characterize those connected height one orders P for which there exists $x \in \text{Max}(P)$ and $y \in \text{Min}(P)$ such that $P - x \simeq P - y$.*

Some families of such orders can be constructed from the family of the n -fence orders. Recall that an n -fence order is any order isomorphic to F_n the connected height one order having for minimal elements the set $\{a_1, \dots, a_n\}$, for maximal elements the set $\{b_1, \dots, b_n\}$, and whose comparabilities are (a_i, b_i) for $i \in \{1, \dots, n\}$ and (a_{i+1}, b_i) for $i \in \{1, \dots, n-1\}$: see Fig. 3(b) for a 2-fence order where φ is an isomorphism onto F_2 with $\varphi(a) = a_1$, $\varphi(b) = a_2$, $\varphi(c) = b_1$ and $\varphi(d) = b_2$. Then, for every $n \geq 1$, we have that $F_n - b_1$ is isomorphic to $F_n - a_n$. But also, adding to F_n the comparability (a_n, b_1) we obtain a family of orders G_n such that $G_n - b_1$ is isomorphic to $G_n - a_n$. Another family obtained that way is the family of orders H_n obtained from F_n by adding the comparabilities (a_i, b_{j_i}) for $j_i \in \{1, \dots, i-2\}$ and for $i \in \{1, \dots, n\}$. Again we have that $H_n - b_1$ is isomorphic to $H_n - a_n$. Also notice that the family of the n -fence orders can be used to obtain families of orders with arbitrary height. Indeed, for any $k \geq 1$, changing in F_n each chain (a_i, b_i) by a chain $(a_i = c_i^0, \dots, c_i^k = b_i)$ we obtain an height k order F_n^k such that $F_n^k - b_1$ is isomorphic to $F_n^k - a_n$.

7.5. Trees and orders

Tree structures are often used to characterize order classes. Some of them, like in the Gallai decomposition, are used to characterize orders by considering structured partitions of their element sets. Others are used to characterize orders either by considering their transitive reductions, or by considering their covering graphs. Here we are interested in this latter approach. Considering the transitive reduction leads to the class of *rooted-tree orders*, that is of orders whose transitive reduction is a rooted directed tree. Equivalently, it is the class of orders whose comparability graph is both connected and such that every path of length (number of vertices) at least 4 has a chord. Since such orders have a least or a greatest element, they are reconstructible (see Section 6.1). Considering the covering graph leads to the class of *tree-like orders*, that is of orders whose covering graph is a tree. This class has been shown to be set-reconstructible (see Section 8.5 for more details on this type of reconstruction) in 1981 by Das [12], but the proof is lengthy (see the comments of Sands recorded in Section 1.8.3), and still unpublished at this time.

For the Ulam reconstruction, we get an easy way to establish the recognition of tree-like orders by looking at the difference between the number of covering edges of an order and the number of covering edges in some of its one-element deleted sub-orders.

Proposition 52. *Let P be a tree-like order. Then there exists $x \in V(P)$ such that $P - x$ is a tree-like order and $|E_C(P)| - |E_C(P - x)| = 1$.*

Proof. Take for x any pendant vertex of the covering graph of P . \square

Proposition 53. *Let P be a connected non-tree-like order, then for every $x \in V(P)$ such that $P - x$ is a tree-like order we have that $|E_C(P)| - |E_C(P - x)| \geq 2$.*

Proof. We begin with some general remarks about the covering graph of an order and the covering graph of its one-element-deleted sub-orders (see Section 2 for the corresponding definitions). For any order Q and for every $x \in V(Q)$ we have that: (i) $E_C(Q - x) = E_C(Q) - \{\{x, y\} : y \in \uparrow_Q^{\text{im}} x\} - \{\{z, x\} : z \in \downarrow_Q^{\text{im}} x\} \cup \text{New}E_C(Q - x)$ where $\text{New}E_C(Q - x) = E_C(Q - x) - E_C(Q)$, (ii) $\text{New}E_C(Q - x) \subseteq \{\{z, y\} : z \in \downarrow_Q^{\text{im}} x, y \in \uparrow_Q^{\text{im}} x\}$, and (iii) the three sets $\{\{x, y\} : y \in \uparrow_Q^{\text{im}} x\}$, $\{\{z, x\} : z \in \downarrow_Q^{\text{im}} x\}$ and $\text{New}E_C(Q - x)$ are pairwise disjoint. We thus have to show that given a connected non-tree-like order P , then for every $x \in V(P)$ such that $P - x$ is a tree-like order, we have that

$$|\text{New}E_C(P - x)| \leq |\{\{x, y\} : y \in \uparrow_P^{\text{im}} x\}| + |\{\{z, x\} : z \in \downarrow_P^{\text{im}} x\}| - 2,$$

that is, that $|\text{New}E_C(P - x)| \leq |\uparrow_P^{\text{im}} x| + |\downarrow_P^{\text{im}} x| - 2$.

Notice that $\text{New}E_C(P - x)$ is the edge set of the subgraph of $C(P - x)$ induced by $\uparrow_P^{\text{im}} x \cup \downarrow_P^{\text{im}} x$, and recall that the subgraph induced by any subset of the set of elements of a tree is a forest, that is, a disjoint union of trees. Consequently we have that $|\text{New}E_C(P - x)| \leq |\uparrow_P^{\text{im}} x| + |\downarrow_P^{\text{im}} x| - 1$, and that moreover the equality holds if and only if the subgraph of $C(P - x)$ induced by $\uparrow_P^{\text{im}} x \cup \downarrow_P^{\text{im}} x$ is connected. We now conclude by contradiction: assume that the subgraph of $C(P - x)$ induced by $\uparrow_P^{\text{im}} x \cup \downarrow_P^{\text{im}} x$ is connected. Consequently, there exists a path in $C(P - x)$ between every pair $\{y, z\} \subseteq (\uparrow_P^{\text{im}} x \cup \downarrow_P^{\text{im}} x)$, say $ch(y, z)$, and whose edges all belong to $E_C(P - x) - E_C(P)$. But, as both P is a connected non-tree-like order and $P - x$ is a tree-like order, then there exists an elementary cycle in $C(P)$ containing x (whose length is greater or equal to four). Let $(x = c_1, \dots, c_k, c_{k+1} = x)$ be such a cycle. By definition of $C(P)$ we have that $c_2 \neq c_k$ and that $c_2, c_k \in (\uparrow_P^{\text{im}} x \cup \downarrow_P^{\text{im}} x)$. Since this cycle is elementary there is no edge, belonging to $\{\{x, y\} : y \in \uparrow_Q^{\text{im}} x\} \cup \{\{z, x\} : z \in \downarrow_Q^{\text{im}} x\}$, other than $\{c_1, c_2\}$ and $\{c_1, c_k\}$. Consequently, with the two edge disjoint paths of $C(P - x)$ between c_2 and c_k , namely $ch(c_2, c_k)$ and the one obtained from $(x = c_1, \dots, c_k, c_{k+1} = x)$ by deleting $x = c_1 = c_{k+1}$, we obtain a cycle. This contradicts the fact that $P - x$ is a tree-like order. \square

Theorem 54. *Tree-like orders are recognizable.*

Proof. By Theorem 25 the number of covering edges of an order is reconstructible. As connected orders are recognizable (see Theorem 30), the result then directly follows from Propositions 52 and 53. \square

Question 55. *Is there a non-lengthy proof that tree-like orders are reconstructible?*

7.6. Truncated lattices

Recall that a finite order T is a truncated lattice if there exists a lattice L isomorphic to \widehat{T} , where \widehat{T} is the order obtained by adding to T both a least element \perp , and a greatest element \top . Recently, in 2000, Rampon and Schröder show that truncated lattices are recognizable.

Theorem 56 (Rampon and Schröder [47]). *Truncated lattices are recognizable.*

Proof (Hint). A truncated lattice T has at least $|\text{Max}(T) \cup \text{Min}(T)|$ cards which are truncated lattices. An order P which is not a truncated lattice has at least $|V(P)| - 4$ cards which are not truncated lattices. Thus it remains to consider the deck of a truncated lattice T with $|\text{Max}(T)| = |\text{Min}(T)| = 2$ and such that for every $x \in V(T) - (\text{Max}(T) \cup \text{Min}(T))$ the card $T - x$ is not a truncated lattice. Then the only non-obvious case is when $h(T) \geq 2$. But the fact, that $T - x$ is not a truncated lattice for every x with rank $h(T) - 1$, implies that the two maximal elements of T have the same rank, and also that T has a unique element with rank $h(T) - 1$. Thus, in this case, T is co-disconnected and even reconstructible (see Theorem 31). \square

Truncated lattices which contain a 4-crown as sub-order are reconstructible as indicated in Section 6.6. Thus it seems that there remains only an easy class of truncated lattices to study. But this class contains the class of height one truncated lattices which is, from the reconstruction point of view, very close to the class of height one orders.

Question 57. *Are truncated lattices which do not contain any 4-crown as sub-order reconstructible?*

8. Other reconstructions of orders

8.1. The $\{-1, 2\}$ -reconstruction of orders

In 1994 Kratsch and Rampon ask the following question: If Q is a reconstruction of P that is not isomorphic to P , does this imply $G(P) \neq G(Q)$? When taking this inequality as inequality of the binary relations a positive answer has been given in 1998 by Ille and Rampon. Indeed they showed that orders are $\{-1, 2\}$ -reconstructible, where two orders P and Q are $\{2\}$ -hypomorphic if and only if $G(P) = G(Q)$. When the inequality is taken as non-isomorphism of the binary relations, the question is still open.

Question 58 (Kratsch and Rampon [32]). *Let P and Q be two $\{-1\}$ -hypomorphic orders such that $G(P) \simeq G(Q)$. Do we have $P \simeq Q$?*

Theorem 59 (Ille, Rampon [28]). *An order P , with $|V(P)| \geq 4$, is $\{-1, 2\}$ -reconstructible.*

Proof (Hint). Assuming that $P = Q(P_x; x \in V(Q))$ and $P' = Q'(P'_x; x \in V(Q'))$ are two $\{-1, 2\}$ -hypomorphic lexicographical sums where Q and Q' are indecomposable, the proof can be done by induction on $|V(P)| \geq 4$. Two cases are then to be considered.

Case 1: $Q = Q'$. Then, either P has at least two non-trivial intervals, and thus, by Lemma 47, we have for all $x \in V(Q)$ $P_x \simeq P'_x$, which implies that $P \simeq P'$. Otherwise, P has a unique non-trivial interval, say S_x , and thus P' has a unique non-trivial interval, say S'_x (see Proposition 11). If $|S_x| = 2$, then $S_x \simeq S'_x$ because $G(P) = G(P')$. If $|S_x| = 3$, the only difficulty would arise when S_x has exactly two comparabilities, and $S_x^d \simeq S'_x$. But Kelly's Lemma forbids such a possibility. Thus we always have $S_x \simeq S'_x$. For $|S_x| \geq 4$, by the induction hypothesis we have $S_x \simeq S'_x$.

Case 2: $Q^d = Q'$. Then P has at least two non-trivial intervals, one containing one of its minimal elements, and the other containing one of its maximal elements. Indeed by contradiction, we have that $\text{Max}(P) = \text{Max}(Q)$. Thus, for $x \in \text{Max}(P)$ with a minimal ideal size among the maximal elements of P , we have that $x \in \text{Min}(P')$ with a minimal filter size among the minimal elements of P' . Recall that the ideal size sequence of the minimal elements and the filter size sequence of maximal elements are reconstructible (see Section 5.2). Then, as $P - x \simeq P' - x$, we obtain that x has a unique immediate predecessor in P , say y , and that moreover y has a unique successor in P . This gives a non-trivial interval of P containing x . We can now conclude that $Q^d = Q'$ is impossible. Indeed, denote by A^+ , respectively A^- , the set of elements x of Q such that $\text{rank}_Q(x) > \text{rank}_{Q^d}(x)$, respectively $\text{rank}_Q(x) < \text{rank}_{Q^d}(x)$. Let $M = \max\{|V(P_x)| \geq 2 : x \in A^+ \cup A^-\}$, and let $a^+ = |\{x : |V(P_x)| = M, x \in A^+\}|$ and $a^- = |\{x : |V(P_x)| = M, x \in A^-\}|$. Note that $a^+ = 0$ if and only if $a^- = 0$, which is impossible. Thus for $x \in A^+$, respectively $x \in A^-$, with $|V(P_x)| = M$, the fact that $P - x \simeq P' - x$ implies that $a^+ - 1 = a^-$, respectively $a^- - 1 = a^+$, which is impossible. \square

The next two results are then deduced from Lemma 1 and from Theorem 59.

Corollary 60 (Hagendorf [22]). *An order P , with $|V(P)| \geq 4$, is $\{2, 3\}$ -reconstructible.*

Proof (Hint). By induction on $|V(P)|$ with Theorem 59. \square

Corollary 61 (Ille and Rampon [28]). *Given $k \geq 2$, an order P , with $|V(P)| \geq k + 3$, is $\{-k\}$ -reconstructible.*

Proof (Hint). The case $k = 2$ is obtained by Lemma 1 and Theorem 59. The case $k \geq 3$ follows from Lemma 1 and Corollary 60. \square

8.2. The max-reconstruction of orders

In 1985, Sands [48] asked the following question: is every finite order P uniquely determined—up to isomorphism—by the family $(P - x)_{x \in \text{Max}(P)}$? In 1994, Kratsch and Rampon give a negative answer to that question, with families of orders of height at least two, and having exactly two maximal elements. Thus, they leave open the case of height one orders, and also the case of orders with an arbitrarily large number of maximal elements. Recently in 1999, Ille and Rampon came up with counterexamples having an arbitrarily large number of maximal elements. As the max-reconstruction is far from completely explored,

and has some interesting extensions, we take some space to present the corresponding notions.

- (i) The *max-deck* of P is the family $(P - x)_{x \in \text{Max}(P)}$ of unlabeled one-element-deleted sub-orders. Each $P - x$ with $x \in \text{Max}(P)$ is said to be a *max-card*.
- (ii) An order Q is a *max-reconstruction* of an order P if there exists a bijection φ from $\text{Max}(P)$ onto $\text{Max}(Q)$ such that, for every $x \in \text{Max}(P)$, $P - x \simeq Q - \varphi(x)$, or, equivalently, if they have the same max-deck.
- (iii) P is said to be *max-reconstructible* if every max-reconstruction of P , or, equivalently, every order with the same max-deck as P , is isomorphic to P .
- (iv) A parameter (on all orders) is said to be *max-reconstructible* if it gives the same value for all max-reconstructions.
- (v) A class \mathcal{C} of orders is said to be *max-recognizable* if all the max-reconstructions of any of its elements belong to \mathcal{C} .

Remarks 62. From the definition of the max-reconstruction we get:

- (i) The number of elements is max-reconstructible.
- (ii) The number of maximal elements is max-reconstructible.
- (iii) Orders with a greatest element are max-reconstructible.

Thus, in the following we only consider orders with at least two maximal elements.

Lemma 63. *The numbers $i(Q, P)$ and $i_{\max}(Q, P)$, of ideal-orders of P isomorphic to Q , and of ideal-orders of the maximal elements of P isomorphic to Q , are max-reconstructible.*

Proof. Let P be an order with $k \geq 2$ maximal elements, and whose decreasing ideal-size sequence of its maximal elements is $i_1 \geq i_2 \geq \dots \geq i_k$. By looking at the decreasing ideal-size sequence of the maximal elements of all the max-cards, we easily obtain that: (i) the first $k - 1$ entries of the decreasing ideal-size sequence of the maximal elements of P are max-reconstructible, and (ii) the fact that all maximal elements of P have the same ideal-size is max-reconstructible. Indeed, (i) comes from the fact that a max-card of P with the greatest first $k - 1$ entries in the decreasing ideal-size of its maximal elements, is obtained in every reconstruction of P by the deletion of a maximal element with smallest ideal-size. For (ii), it comes from the fact that all the maximal elements of P have the same ideal-size if and only if all the first $k - 1$ entries of the decreasing ideal-size sequence of the maximal elements of all the max-cards of P are identical. We continue with a simple case analysis on the ideal-size sequence of the maximal elements of P .

Case 1: all the maximal elements of P have an ideal-size equal to s . Then for $|V(Q)| = s$, we have $i_{\max}(Q, P) \cdot (k - 1) = \sum_{x \in \text{Max}(P)} i_{\max}(Q, P - x)$, and for $|V(Q)| \neq s$, we have $i_{\max}(Q, P) = 0$. Now, from any max-card for any Q we obtain $i(Q, P)$.

Case 2: the maximal elements of P do not have the same ideal size, and thus the number of maximal elements of P , having i_1 for ideal-size, is max-reconstructible. Let $P - x$ be any max-card having $i_1 \geq i_2 \geq \dots \geq i_{k-1} \geq i'_k \geq \dots \geq i'_l$ for ideal-size sequence of its maximal elements. Let $X \subseteq \text{Max}(P - x)$ be the set $\{z : z \in \text{Max}(P - x) \text{ and } |\downarrow_{P-x}^1 z| = i_1\}$. Let $P - y$ be any max-card which does not have the same number of maximal elements

having i_1 for ideal-size as P does. Let $Y \subseteq \text{Max}(P - y)$ be the set $\{z : z \in \text{Max}(P - y) \text{ and } |\downarrow_{P-y}^1 z| = i_1\}$. Then, the ideal-order induced by the missing element of $P - x$ is isomorphic, in any reconstruction of P , to the order Z , where Z is the unique order Q such that $i_{\max}(Q, P - y[V(P - y) - Y]) - i_{\max}(Q, P - x[V(P - x) - X]) > 0$. Consequently, from $P - x$, we easily obtain that $i_{\max}(Q, P)$ and $i(Q, P)$ are reconstructible.

Proposition 64. *Let P be an order. Then the following parameters are max-reconstructible:*

- (i) *Its height $h(P)$ and its width $w(P)$.*
- (ii) *The ideal-size sequence of its maximal elements.*
- (iii) *The number of edges of its covering graph.*
- (iv) *The number of edges of its comparability graph $|E(G(P))|$.*

Proof. Note that for any order Q , we have $w(Q) = \max\{|\text{Max}(Q)|, \max\{w(Q - x) : x \in \text{Max}(Q)\}\}$, and, for $|\text{Max}(Q)| \geq 2$, we have $h(Q) = \max\{h(Q - x) : x \in \text{Max}(Q)\}$. Conditions (ii) to (iv) follow from Lemma 63.

8.2.1. Some max-reconstructions

Interval order: Recall that an order P is an interval order if it can be represented by assigning a real interval $I_x = [l(x), r(x)]$ to each element $x \in V(P)$, such that $x <_P y$ if and only if $r(x) <_{\mathbb{R}} l(y)$ for all $x, y \in V(P)$. See Section 6.3 for a characterization theorem of such orders.

Theorem 65. *Interval orders are max-reconstructible.*

Proof. Recall that we only consider orders with at least two maximal elements. Note that any interval order P has at least one max-card having exactly $|\text{Max}(P)| - 1$ maximal elements. Note that given an order Q all of whose max-cards are interval orders, then Q is not an interval order if and only if (i) $|\text{Max}(Q)| = 2$, and (ii) each of its maximal elements has a private predecessor (an immediate predecessor having a unique immediate successor). Thus, the number of maximal elements in any max-card of Q is at least two. This insures that interval orders are max-recognizable. To show that interval orders are max-reconstructible, let P be an interval order, and let $P - x$ be a max-card, where the deleted element is of maximal ideal-size. Recall that, from Proposition 64 (ii), the ideal-size sequence of the maximal elements is max-reconstructible. Let Z be any subset of $\text{Max}(P - x)$ such that the ideal-size sequence of the elements of Z is exactly the ideal-size sequence of the maximal elements of P minus the ideal-size of x . From Theorem 34 (iii), it then follows that (i) any max-reconstruction of P is obtained from $P - x$ by adding an element z having for predecessor set $\downarrow_{P-x}^1 (\text{Max}(P - x) - Z) \cup \downarrow_{P-x}^{\text{im}} Z$, and that (ii) all such orders are isomorphic.

Width two orders: As indicated in Section 6.4 width two orders are natural candidates to be max-reconstructible. Indeed, the proof can use the same skeleton as the one given for the Ulam reconstruction.

Theorem 66. *Width two orders are max-reconstructible.*

Proof. Width two orders are max-recognizable by Proposition 64 (i). Let P be a width two order having $i_1 \geq i_2$ for decreasing ideal size sequence of its maximal elements. For any reconstruction Q of P , and thus for P too, denote by x_1, x_2 its maximal elements, and assume that $|\downarrow_Q^1 x_1| = i_1$. Two cases have then to be considered: Case 1: $|\text{Max}(P - x_2)| = 2$ and Case 2: $|\text{Max}(P - x_2)| = 1$. Let P^* be any max-card of P having one element with i_1 for ideal size.

Firstly, assume that $\text{Max}(P^*) = \{y_1, y_2\}$ with $|\downarrow_{P^*}^1 y_1| = i_1$: we are thus in Case 1. Then, every max-reconstruction of P is obtained by adding to P^* a new maximal element, say x , having i_2 for ideal size. As $i_1 \geq i_2$ then x has necessarily y_2 for immediate predecessor. Now, either $|\downarrow_{P^*}^1 y_2| = i_2 - 1$, and thus y_2 is the unique immediate predecessor of x . Or $|\downarrow_{P^*}^1 y_2| < i_2 - 1$, and then x has also, for immediate predecessor, the unique element $z \in V(P^*) - \downarrow_{P^*}^1 y_2$ such that $|\downarrow_{P^*}^1 z - \downarrow_{P^*}^1 y_2| = (i_2 - 1) - |\downarrow_{P^*}^1 y_2|$.

Secondly, assume that $\text{Max}(P^*) = \{y_1\}$ with $|\downarrow_{P^*}^1 y_1| = i_1$: we are thus in Case 2. Consequently, in any reconstruction Q of P we have that $\downarrow_Q^1 x_2 \subseteq \downarrow_Q^1 x_1$. Now, one possibility is that P^\dagger , the other max-card which is isomorphic to $P - x_1$, has a unique maximal element, and thus in any reconstruction Q of P we have that $\downarrow_Q^1 x_2 = \downarrow_Q^1 x_1$. Otherwise, P^\dagger has two maximal elements, and so we assume that $\text{Max}(P^\dagger) = \{y, z\}$. Then, the only non-direct case, for showing that P is reconstructible, is when both $|\downarrow_{P^\dagger}^1 y| = |\downarrow_{P^\dagger}^1 z| = i_2$ and $\text{rank}_{P^\dagger}(y) = \text{rank}_{P^\dagger}(z) = \text{rank}_P(x_2)$. Now we use the card P^* . Note that the immediate predecessor of the greatest element of P^* , whose rank is distinct from $\text{rank}_P(x_2)$, is also an immediate predecessor of x_2 in P . Then, using the same argumentation as that used in Case 1, we can show that P is max-reconstructible. \square

Height one orders: Up to now, the only non-trivial result in the max-reconstruction of height one orders is the one obtained in 1968 by S. Hyvärö. The original statement was given in terms of properly 2-vertex-colored bipartite graphs. Recall that a k -vertex-colored graph is a pair (G, φ) , where G is a graph and φ is a k -coloring of its vertices. The pair (G, φ) is a properly k -vertex-colored graph if $\varphi(x) \neq \varphi(y)$ whenever yx is an edge in G . Two vertex-colored graphs are isomorphic if there is an isomorphism between the graphs preserving the color classes. S. Hyvärö's result is based on the following straightforward property reducing the test of isomorphism between two height one orders to comparisons of cardinals.

Proposition 67. *Let P and Q be two height one orders. P is isomorphic to Q if and only if there exists a bijection φ from $\text{Max}(P)$ onto $\text{Max}(Q)$ such that for all $X \subseteq \text{Max}(P)$ we have that $|\{y \in \text{Min}(P) : \uparrow_P^1 y = X\}| = |\{y \in \text{Min}(Q) : \uparrow_Q^1 y = \varphi(X)\}|$. Here φ is extended to a subset Z of $\text{Max}(P)$ by $\varphi(Z) = \bigcup_{z \in Z} \varphi(z)$.*

Theorem 68 (Hyvärö [25]). *Let P be a height one order, and let (d_1, d_2, \dots, d_k) be the ideal-size sequence of its maximal elements. Then, P is max-reconstructible if one of the following conditions holds:*

- (i) For all $1 \leq i < j \leq k$, $d_i \neq d_j$

- (ii) $d_1 = d_2$ and for all $2 \leq i < j \leq k$, $d_i \neq d_j$
- (iii) $d_1 = d_2 = d_3$ and for all $3 \leq i < j \leq k$, $d_i \neq d_j$

Proof (Hint). Let P be a height one order. If $\text{Max}(P) \cap \text{Min}(P) \neq \emptyset$, P is clearly max-reconstructible. Thus, assume that P has no isolated element. From Proposition 67, it is sufficient to show that for every $X \subseteq \text{Max}(P)$, the number $|\{y \in \text{Min}(P) : \uparrow_P^l y = X\}|$ is max-reconstructible. The result follows from the fact that (a) for $X \subseteq \text{Max}(P)$, $|\{y \in \text{Min}(P) : \uparrow_P^l y = X\}|$ is the number of isolated elements in $P - X$ minus the number of isolated elements in $P - (X - z)$ for any $z \in X$, and that (b) for every $x \in \text{Max}(P)$, $|\{y \in \text{Min}(P) : \uparrow_P^l y = x\}|$ is the number of isolated elements in $P - x$. Denote by x_i the element of $\text{Max}(P)$ having d_i for ideal-size.

Case (i) is now straightforward since, in each card, the maximal elements of P are uniquely determined by their ideal size.

For Case (ii) choose arbitrarily, among the two possible choices, one card for x_1 and the other one for x_2 .

For Case (iii) choose arbitrarily, among the three possible choices, one card for x_1 , one card for x_2 , and the remaining one for x_3 . Now, the only difficulty is to distinguish the subsets of $\text{Max}(P)$ containing a given two-element subset of $\{x_1, x_2, x_3\}$. But the distinction can be done by a simple and careful analysis of the isomorphism type, of the orders $P - \{x_i, z_i\}$ and $P - \{x_i, t_i\}$, for $i \in \{1, 2, 3\}$, and for z_i, t_i having, in $P - x_i$, d_1 for ideal-size. \square

8.2.2. Counterexamples to the max-reconstruction

In 1994 Kratsch and Rampon [31] answer negatively the question of Sands on the max-reconstruction, by pairs of infinite families of orders, having two maximal elements, and being of height at least two. All these pairs are constructed from a pair of two nine-element orders. This pair of orders also provided a negative answer to the following second question of Sands: Is every finite order P uniquely determined—up to isomorphism—by the family $(P - \uparrow_P^l x)_{x \in V(P)}$? However, as was noticed by Mittas, one of the nine elements is not necessary. Thus we give in Fig. 7 this new pair of orders having only the necessary elements.

Remarks 69. Directly from both the above counterexample and the counterexample of Schröder given in Fig. 5 of Section 6.4, we can deduce some non-max-reconstructible classes of orders and parameters. Note that, adding a least element to P_1 and a least element to P_2 , we obtain a pair of co-disconnected orders that are not max-reconstructible.

- (i) Decomposable orders are not max-reconstructible.
- (ii) Co-disconnected orders are not max-reconstructible.
- (iii) Width k orders are not max-reconstructible, for $k \geq 3$.
- (iv) Height k orders are not max-reconstructible, for $k \geq 2$.
- (v) Orders of dimension two are not max-reconstructible.
- (vi) The filter-size sequence is not max-reconstructible.

The fact that all the counterexamples have two maximal elements implies that the families of max-cards have two elements. This is not completely satisfactory for a general counterexample in reconstruction. Thus, in 1999, Ille and Rampon [29] strengthen this result with

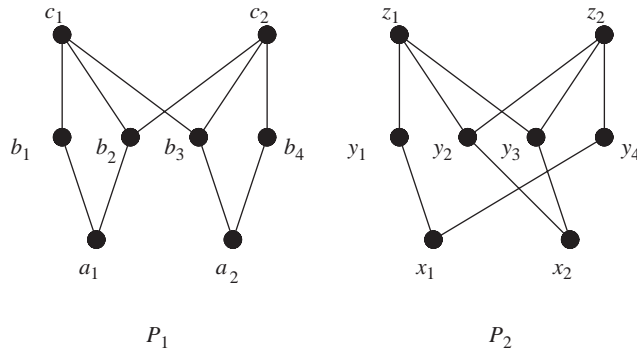


Fig. 7. On the one hand $P_1 - \alpha \simeq P_2 - \beta$ for $\alpha \in \{c_1, c_2\}$ and $\beta \in \{z_1, z_2\}$. On the other hand P_1 and P_2 are not isomorphic. Indeed, P_1 has only trivial intervals, and P_2 has also $\{y_2, y_3\}$ for interval. Note that P_1 is of dimension two and admits $\{(a_1, b_1, b_2, a_2, b_3, c_1, b_4, c_2), (a_2, b_4, b_3, a_1, b_2, c_2, b_1, c_1)\}$ for realizer.

an infinite family of orders having an arbitrarily great number of maximal elements. A pair of counterexamples with $k + 1$ maximal elements, say (P_1, P_2) , is obtained from the pair of counterexamples with k maximal elements, say (Q_1, Q_2) , in the following way. Let φ be the one-to-one mapping, from $\text{Max}(Q_1)$ onto $\text{Max}(Q_2)$, such that for every $x \in \text{Max}(Q_1)$ we have that $Q_1 - x$ is isomorphic to $Q_2 - \varphi(x)$ by ψ_x . Let Z be the order such that: (i) $V(Z)$ is the disjoint union of $V(Q_1)$ and $V(Q_2) - \text{Max}(Q_2)$, and (ii) $Z[V(Q_1)] = Q_1$, $Z[V(Q_2) - \text{Max}(Q_2)] = Q_2[V(Q_2) - \text{Max}(Q_2)]$, and for every $x \in \text{Max}(Q_1)$ we have $\downarrow_Z^1 x = \downarrow_{Q_1}^1 x \cup \downarrow_{Q_2}^1 \varphi(x)$. Then P_1 is obtained from Z by adding a new maximal element whose predecessor set is $V(Q_1) - \text{Max}(Q_1)$, and P_2 is obtained from Z by adding a new maximal element whose predecessor set is $V(Q_2) - \text{Max}(Q_2)$. The two orders P_1 and P_2 are not isomorphic simply because Q_1 and Q_2 are not isomorphic. The two orders P_1 and P_2 have the same max-deck because for every $x \in \text{Max}(Q_1) - x$, $\psi_x(y) = \varphi(y)$. An example for (P_1, P_2) with 3 maximal elements is given in Fig. 8.

8.2.3. Open questions

Natural candidates for the max-reconstruction are classes of orders which are not known to be Ulam reconstructible. Particularly we have the classes of height one orders that we discuss in Section 8.2.1. Among the results we already have for the Ulam reconstruction, it seems interesting to study the status of the following parameters, and of the classes of orders which are directly related to them.

Question 70. *Are disconnected orders max-recognizable?*

Question 71. *What about the max-reconstruction of disconnected orders, and series-parallel orders?*

Some extensions of Sands' question are investigated in Section 8.3 and in Section 8.4.

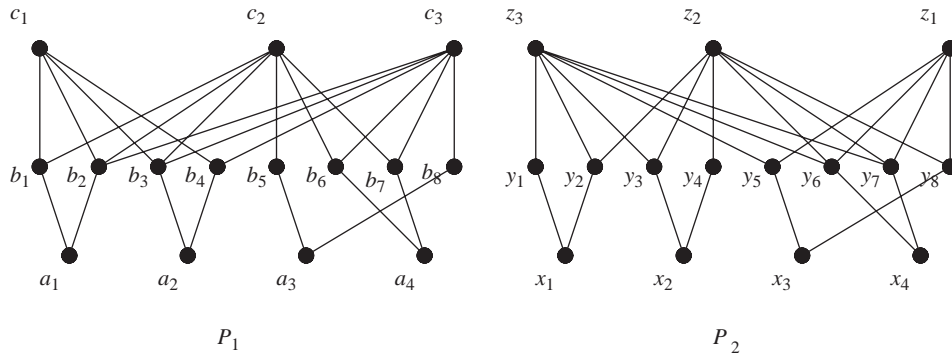


Fig. 8. On the one hand $P_1 - c_1 \simeq P_2 - z_1$ and $P_1 - \alpha \simeq P_2 - \beta$ for $\alpha \in \{c_2, c_3\}$ and $\beta \in \{z_2, z_3\}$. On the other hand P_1 and P_2 are not isomorphic. Indeed, the elements of P_2 having $\text{Max}(P_2)$ for immediate successors are y_6 and y_7 , and the elements of P_1 having $\text{Max}(P_1)$ for immediate successors are b_2 and b_3 . Then notice that $\{y_6, y_7\}$ is an autonomous set in P_2 , and that $\{b_2, b_3\}$ is not an autonomous set in P_1 .

8.3. The min–max-reconstruction of orders

A natural extension of Sands' question is to consider, for the family of one-element-deleted sub-orders, those obtained from both the minimal and the maximal elements. This led us in 1996, see [46], to ask for the min–max-reconstruction of orders: Is every finite order P , on at least five elements, uniquely determined—up to isomorphism—by the family $(P - x)_{x \in \text{Min}(P) \cup \text{Max}(P)}$?

Recently, at the very end of 2001, an interesting paper of Schröder [52] provides a negative answer to this min–max-reconstruction of orders. Indeed, in his paper, Schröder exhibits families of counterexamples with arbitrarily large numbers of maximal and minimal elements, obtained with a gluing method similar to the one of Ille and Rampon for the max-reconstruction. Schröder introduces an original and interesting method to obtain the starting pairs of counterexamples with two maximal and two minimal elements. We report one of the smallest pairs of counterexamples in Fig. 9. On the one hand, to see that P_1 and P_2 are not isomorphic, notice that any isomorphism φ from P_1 to P_2 would be such that $\varphi(c) = c$ and $\varphi(d) = d$, and consequently we would have $\varphi(\{A, B\}) = \{A, B\}$. But this is not compatible with the fact that (i) there are two covering relations in P_1 between A and $\{a, b\}$, and (ii) there are three covering relations in P_2 both between A and $\{a, b\}$ and between B and $\{a, b\}$. On the other hand, for every $x \in \{a, b, c, d\}$, we can construct an isomorphism, say φ_x , from $P_1 - x$ on to $P_2 - x$. To do so simply notice that:

- (1) for φ_a , we have $\varphi_a(A) = B$, $\varphi_a(B) = A$, and $\varphi_a(C) = D$.
- (2) for φ_b , we have $\varphi_b(A) = A$, $\varphi_b(B) = B$, and $\varphi_b(C) = C$.
- (3) for φ_c , we have $\varphi_c(A) = C$, $\varphi_c(B) = D$, and $\varphi_c(C) = B$.
- (4) for φ_d , we have $\varphi_d(A) = C$, $\varphi_d(B) = D$, and $\varphi_d(C) = A$.

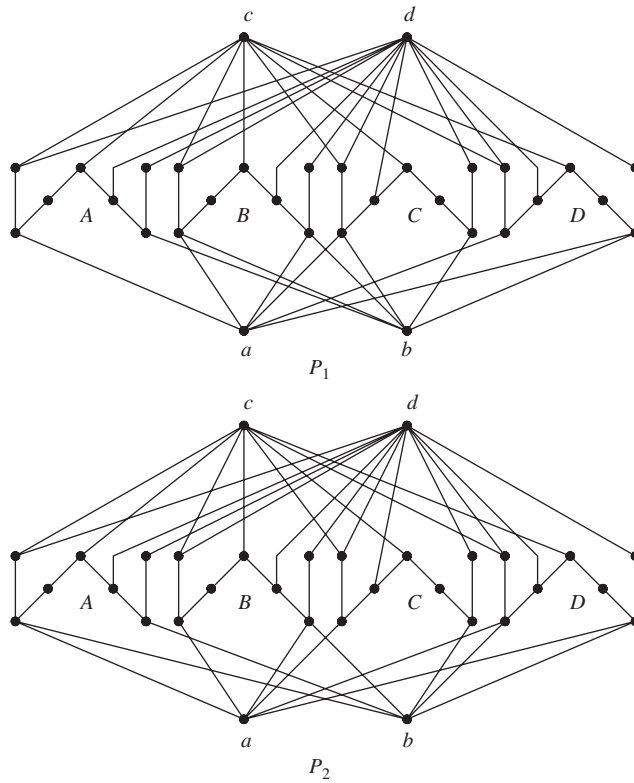


Fig. 9. Two non-isomorphic orders such that $P_1 - x \simeq P_2 - x$ for any $x \in \{a, b, c, d\}$.

8.4. The inner-reconstruction of orders

The max-reconstruction admits a negative answer and thus provides also a negative answer for all reconstructions, using a deck restricted to one level of an order, that is, when the family of the one-element-deleted sub-orders is constructed from any fixed level of the order. Indeed, for any $k \geq 0$, from either the two orders P_1 and P_2 of Fig. 7, or from their dual orders, and by the series composition with two appropriate total orders (one for below and one for above), we can obtain a new pair of non-isomorphic orders having the same—up to isomorphism—family of one-element-deleted sub-orders obtained from their level k . However, nothing can be deduced from the max-reconstruction when the family of one-element-deleted sub-orders comes from all the elements of the order except the minimal and maximal ones. More formally, the *inner-deck* of an order P is the family $(P - x)_{x \in V(P) - (\text{Min}(P) \cup \text{Max}(P))}$ of unlabeled one-element-deleted sub-orders. The inner-reconstruction thus appears as complementary to the min-max-reconstruction. In the case of height two orders we can show that this inner-reconstruction admits a negative answer by using a result on the hypergraph reconstruction.

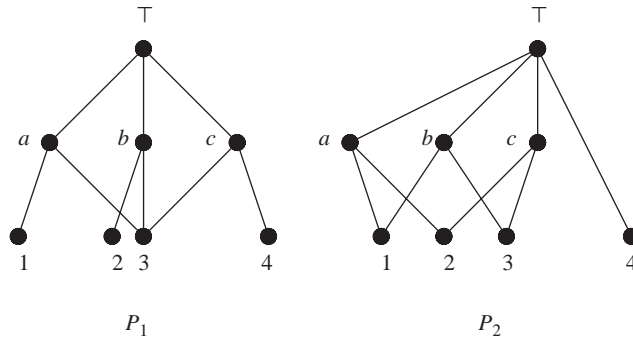


Fig. 10. Two non-isomorphic orders such that $P_1 - x \simeq P_2 - x$ for any $x \in \{a, b, c\}$.

In 1972 Berge and Rado showed that hypergraphs were not Ulam reconstructible. Recall that a hypergraph is a family of non-empty sets, that is, a pair $\mathcal{H} = (X, \mathcal{E})$, where $\mathcal{E} = (E_i)_{i \in I}$, $\bigcup_{i \in I} E_i = X$, and $E_i \neq \emptyset$ for every $i \in I$. Two hypergraphs $\mathcal{H}_1 = (X_1, (E_i^1)_{i \in I_1})$ and $\mathcal{H}_2 = (X_2, (E_i^2)_{i \in I_2})$ are said to be isomorphic if there exists (φ, σ) a pair of one to one mappings, φ from X_1 onto X_2 , and σ from I_1 onto I_2 , such that for every $i \in I_1$ we have $\varphi(E_i^1) = E_{\sigma(i)}^2$, where φ is extended to a subset Z of X_1 by $\varphi(Z) = \bigcup_{z \in Z} \varphi(z)$.

Lemma 72 (Berge and Rado [2]). *Let X be any finite set. Let $\mathcal{H}_1 = (X, \mathcal{E}_1)$ where \mathcal{E}_1 is any ordering of the set $\{Z : Z \subseteq X, |Z| = 2k + 1 \text{ for } k \in \mathbb{N}\}$. Let $\mathcal{H}_2 = (X, \mathcal{E}_2)$, where \mathcal{E}_2 is any ordering of the set $\{Z : \emptyset \neq Z \subseteq X, |Z| = 2k \text{ for } k \in \mathbb{N}\}$. Then the hypergraphs \mathcal{H}_1 and \mathcal{H}_2 are $\{-1\}$ -hypomorphic, and are not isomorphic.*

Proof (Hint). Note that $|I_1| \neq |I_2|$, and that, for any $x \in X$, we have $\{\emptyset \neq Z - \{x\} : Z \subseteq X, |Z| = 2k + 1 \text{ for } k \in \mathbb{N}\} = \{Z - \{x\} : \emptyset \neq Z \subseteq X, |Z| = 2k \text{ for } k \in \mathbb{N}\}$. \square

From that counterexample we can immediately deduce that:

Proposition 73. *Orders are not inner-reconstructible.*

Proof. Let X be any finite set, and let \mathcal{H}_1 and \mathcal{H}_2 be the two hypergraphs on X as defined in Lemma 72. To \mathcal{H}_1 we associate the order P_1 , where $V(P_1)$ is the disjoint union of the three sets X , I_1 and $\{\top\}$, and whose covering edges are exactly: for every $x \in X$, $x \prec_{P_1} \top$; for every $i \in I_1$, and for every $x \in E_i^1$, $i \prec_{P_1} x$. To \mathcal{H}_2 we associate the order P_2 , where $V(P_2)$ is the disjoint union of the four sets X , I_2 , $\{|I_2| + 1\}$, and $\{\top\}$, and whose covering edges are exactly: for every $x \in X$, $x \prec_{P_2} \top$; for every $i \in I_2$, and for every $x \in E_i^2$, $i \prec_{P_2} x$; and $\{|I_2| + 1\} \prec_{P_2} \top$ ($\{|I_2| + 1\}$ corresponds to the empty subset of any power set). Note that P_1 and P_2 are not isomorphic since only P_2 has a minimal element having for immediate successor a maximal element. To conclude the proof we easily deduce from Lemma 72 that for every $x \in X$ we have $P_1 - x \simeq P_2 - x$ (Fig. 10).

Question 74. *Are orders of height at least 3 inner-reconstructible?*

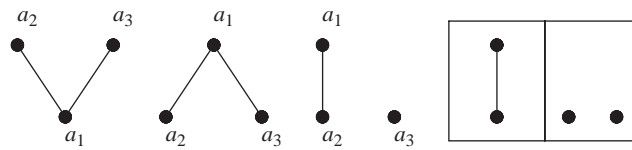


Fig. 11. The \vee -order, the \wedge -order and the $2 \oplus 1$ order: three non-pairwise isomorphic orders with the same set-deck.

8.5. The set reconstruction of orders

Except for the works of Das, the set-reconstruction of orders has not encountered a lot of interest up to now. We think, on the contrary, that it is worthwhile to pay some attention to this kind of reconstruction. Indeed, set-reconstruction requires structural results, on the links existing between an order and the number of isomorphism types of its one-element-deleted sub-orders. We now present the notions corresponding to set-reconstruction.

- (i) The *set-deck* of an order P is the set $\{P - x : x \in V(P)\}$ of unlabeled one-element-deleted sub-orders. Each $P - x$ with $x \in V(P)$ is said to be a *card*.
- (ii) An order Q is a *set-reconstruction* of an order P if both for any $x \in V(P)$ there exists $y \in V(Q)$ such that $P - x \simeq Q - y$, and for any $y \in V(Q)$ there exists $x \in V(P)$ such that $Q - y \simeq P - x$. Or, equivalently, if Q and P have the same set-deck.
- (iii) P is said to be *set-reconstructible*, if every set-reconstruction of P , or, equivalently, every order with the same set-deck as P , is isomorphic to P .
- (iv) A parameter (on all orders) is said to be *set-reconstructible* if it gives the same value for all set-reconstructions.
- (v) A class \mathcal{C} of orders is said to be *set-recognizable* if all the set-reconstructions of any of its elements belong to \mathcal{C} .

Remark 75. As in Ulam's reconstruction, the set-reconstruction question applies only to orders with at least four elements. Thus chain orders and anti-chain orders are clearly set-reconstructible. Note that, in the case of set-reconstruction, two pairs of counterexamples on three elements are now available (see Fig. 11).

As already evoked in Section 5.3, in 1979 Das announced in the paper [10] that he proved, in a submitted paper of 1976, the set-reconstruction of the neighborhood deck of orders. He then used this result to establish the set-reconstruction of a class of orders defined by conditions on the immediate predecessor sets, and by conditions between the number of elements of consecutive ranks.

Theorem 76 (Das [9]). *The neighborhood deck of an order is set-reconstructible.*

As this result has not been published yet, and since it allows us to deduce a lot of interesting parameters (see Section 5.3), we think that it is worth having a short proof for it.

Question 77. Find a short proof for Theorem 76. Partial results, like short proofs of the set-reconstruction of the ideal-orders sequence, the ideal-size sequence, or the level distribution, are also to be considered.

As already evoked in Section 7.5, the set-reconstruction of tree-like orders has been announced by Das in his Ph.D. Thesis [12] in 1981, but is still unpublished at this time.

Theorem 78 (Das [12]). *Tree-orders are set-reconstructible.*

Regarding the number of isomorphism types of the one-element-deleted sub-orders of a given order, recall that the $\{-1\}$ -monomorphic orders are those orders whose one-element-deleted sub-orders are all isomorphic. It is not difficult to see that $\{-1\}$ -monomorphic orders are characterizable as being lexicographical sums of any given total order over any given anti-chain order.

Question 79. Let a $\{-1\}$ -bimorphic order be any order whose one-element-deleted sub-orders have only two isomorphism types. Find a characterization of the class of $\{-1\}$ -bimorphic orders.

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References

- [1] M. Basso-Gerbelli, P. Ille, La reconstruction des relations définies par interdits, C. R. Acad. Sci. Paris, t. 316, Série I, 1993, pp. 1229–1234.
- [2] C. Berge, R. Rado, Note on isomorphic hypergraphs and some extensions of Whitney's theorem to families of sets, J. Combin. Theory Ser. B 13 (1972) 226–241.
- [3] G. Birkhoff, Lattice theory, American Mathematical Society Colloquium Publications, vol. XXV, 1967.
- [4] K.P. Bogart, An obvious proof of Fishburn's interval order theorem, Discrete Math. 118 (1–3) (1993) 239–242.
- [5] J.A. Bondy, A graph reconstructor's manual, Surveys in Combinatorics, in: Proceedings of the 13th British Combinatorial Conference, 1991, pp. 221–252.
- [6] J.A. Bondy, R.L. Hemminger, Graph reconstruction—a survey, J. Graph Theory 1 (1977) 227–268.
- [7] M. Capobianco, J.C. Molluzzo, Examples and counterexamples in graph theory, Foreword by Gary Chartrand, North-Holland, New York–Amsterdam–Oxford, vol. XII, Graph Reconstruction, 1978, pp. 135–152 (Chapter 8).
- [8] S.K. Das, On the structure of finite $T_0 + T_5$ spaces, Canad. J. Math. XXV (6) (1973) 1148–1158.
- [9] S.K. Das, Reconstructible invariants of finite topological spaces, I and II, 1976, submitted for publication.
- [10] S.K. Das, Reconstruction of a class of finite acyclic digraphs, Graph Theory, Proc. Symp. Calcutta 1976, ISI Lecture Notes 4 (1979) 163–173.
- [11] S.K. Das, Set-reconstruction of chain sizes in a class of finite topologies, Combinatorics and graph theory, Proc. Symp. Calcutta 1980, Lecture Notes in Mathematics 885 (1980) 227–236.

- [12] S.K. Das, Some studies in the theory of finite topologies, Ph.D. Thesis, University of Calcutta, 1981.
- [13] B.A. Davey, Problem 10.21, in: I. Rival (Ed.), *Ordered Sets*, D. Reidel, Dordrecht, 1981, p. 859.
- [14] G. Dirac, On rigid circuit graphs, *Abh. Math. Sem. Univ. Hamburg* 25 (1961) 71–76.
- [15] A. El-Issawi, La $\{-1\}$ -reconstruction des tournois $\{-1\}$ -monomorphes, *C. R. Acad. Sci. Paris*, t 322, Série I, 1996, p. 1015–1018.
- [16] P.C. Fishburn, *Interval Orders and Interval Graphs: A Study of Partially Ordered Sets*, A Wiley-Interscience Publication, 1985.
- [17] J. Fisher, R.L. Graham, F. Harary, A simpler counterexample to the reconstruction conjecture for denumerable graphs, *J. Combin. Theory Ser. B* 12 (1972) 203–204.
- [18] R. Fraïssé, Abriement entre relations et spécialement entre chaînes, *Sympos. Math.* 5 (1970) 203–251.
- [19] R. Fraïssé, G. Lopez, La reconstruction d'une relation dans l'hypothèse forte: isomorphie des restrictions à chaque partie stricte de la base, *Séminaire de Mathématiques Supérieures, Séminaire Scientifique OTAN (NATO Advanced Study Institute) no. 109*, Presses de l'Université de Montréal, 1990.
- [20] T. Gallai, Transitiv orientierbare Graphen, *Acta Math. Acad. Sci. Hungar.* 18 (1967) 25–66.
- [21] C. Gnanvo, P. Ille, La restructibilité des tournois, *C. R. Acad. Sci. Paris*, t. 306, Série I, 1988, p. 577–580.
- [22] J.-G. Hagendorf, Restriction respectueuse et reconstruction des chaînes et des relations infinies, *Z. Math. Logik Grundlag. Math.* 38 (1992) 457–490.
- [23] F. Harary, On the reconstruction of a graph from its collection of subgraphs, in: M. Fielder (Ed.), *Theory of Graphs and its Applications*, Prague, 1964, pp. 47–52.
- [24] F. Harary, A survey of the reconstruction conjecture, *Graphs and Combin., Proc. Capital Conf.*, Washington, D.C. 1973, *Lecture Notes in Math.* 406 (1974) 18–28.
- [25] S. Hyvär, Einige Bemerkungen über Rekonstruktion des Graphen aus seinen Untergraphen, *Ann. Univ. Turku Ser. AI* 118 (7) (1968) 7.
- [26] P. Ille, Indecomposable graphs, *Discrete Math.* 173 (1997) 71–78.
- [27] P. Ille, Recognition problem in reconstruction for decomposable relations, in: N.W. Sauer et al. (Eds.), *Finite and Infinite Combinatorics in Sets and Logic*, Kluwer Academic Publishers, 1993, pp. 189–198.
- [28] P. Ille, J.-X. Rampon, Reconstruction of posets with the same comparability graph, *J. Combin. Theory Ser. B* 74 (1998) 368–377.
- [29] P. Ille, J.-X. Rampon, Some counterexamples on reconstructions, 2000, preprint.
- [30] P.J. Kelly, A congruence theorem for trees, *Pacific J. Math.* 7 (1957) 961–968.
- [31] D. Kratsch, J.-X. Rampon, A counterexample about poset reconstruction, *Order* 11 (2) (1994) 95–96.
- [32] D. Kratsch, J.-X. Rampon, Towards the reconstruction of posets, *Order* 11 (4) (1994) 317–341.
- [33] D. Kratsch, J.-X. Rampon, Width two posets are reconstructible, *Discrete Math.* 162 (1996) 305–310.
- [34] D. Kratsch, H. Müller, J.-X. Rampon, About the reconstruction of height one orders, 1998, preprint.
- [35] B. Leclerc, B. Monjardet, Ordres “C.A.C.”, *Fundam. Math.* 79 (1973) 11–22.
- [36] B. Leclerc, B. Monjardet, Corrections à l'article: Ordres “C.A.C.”, *Fundam. Math.* 85 (1974) 97.
- [37] G. Lopez, L'indéformabilité des relations et des multirelations binaires, *Z. Math. Logik Grundlag. Math.* 24 (1978) 303–317.
- [38] G. Lopez, C. Rauzy, Reconstruction of binary relations from their restrictions of cardinality 2,3,4 and $(n-1)$, I, *Z. Math. Logik Grundlag. Math.* 38 (1992) 27–37.
- [39] G. Lopez, C. Rauzy, Reconstruction of binary relations from their restrictions of cardinality 2,3,4 and $(n-1)$, II, *Z. Math. Logik Grundlag. Math.* 38 (1992) 157–168.
- [40] B. Manvel, Reconstructing the degree pair sequence of a digraph, *J. Combin. Theory Series B* 15 (1973) 18–31.
- [41] C.St.J.A. Nash-Williams, The reconstruction problem, *Selected Topics in Graph Theory*, vol. 1, New York, 1978, pp. 205–236.
- [42] J. Nešetřil, On reconstructing of infinite forests, *Commentations Math. Univ. Carolinae* 13 (1972) 503–510.
- [43] M. Pouzet, Application d'une propriété combinatoire des parties d'un ensemble aux groupes et aux relations, *Math Z.* 150 (1976) 117–134.
- [44] M. Pouzet, Note, sur certains tournois reconstructibles, application à leurs groupes d'automorphismes, *Discrete Math.* 24 (1978) 225–229.
- [45] M. Pouzet, Relations non reconstructibles par leurs restrictions, *J. Combin. Theory Series B* 26 (1979) 22–34.
- [46] J.-X. Rampon, Structures et algorithmiques des ensembles ordonnés, *Habilitation à diriger des recherches de l'Université de Rennes I*, 1996.

- [47] J.-X. Rampon, B.S.W. Schröder, Reconstruction of finite truncated semi-modular lattices, *J. Combin. Theory Series B* 86 (2002) 254–272.
- [48] B. Sands, Unsolved Problems, *Order* 1 (1985) 311–313.
- [49] B.S.W. Schröder, Reconstruction of the neighborhood deck of an ordered set, *Order* 17 (3) (2000) 255–269.
- [50] B.S.W. Schröder, *Ordered sets. An Introduction*. Birkhäuser, Boston, 2003.
- [51] B.S.W. Schröder, Reconstruction of N-free ordered sets, *Order* 18 (1) (2001) 61–68.
- [52] B.S.W. Schröder, Examples on ordered set reconstruction. *Order* 19 (2002) 283–294.
- [53] B.S.W. Schröder, On ordered sets with isomorphic marked maximal cards. *Order* 20 (2003) 299–327.
- [54] J.H. Schmerl, W.T. Trotter, Critically indecomposable partially ordered sets, graphs, tournaments and other binary relational structures, *Discrete Math.* 113 (1993) 191–205.
- [55] P.K. Stockmeyer, The falsity of the reconstruction conjecture for tournaments, *J. Graph Theory* 1 (1977) 19–25.
- [56] W.T. Trotter, *Combinatorics and Partially Ordered Sets: Dimension Theory*, The John Hopkins University Press, Baltimore, Maryland, 1992.
- [57] W.T. Trotter, J.I. Moore, Characterization problems for graphs, partially ordered sets, lattices, and families of sets, *Discrete Math.* 16 (1976) 361–381.
- [58] S.M. Ulam, *A Collection of Mathematical Problems*, Interscience Publisher, New York, 1960.
- [59] M. Von Rimscha, Reconstructibility and perfect graphs, *Discrete Math.* 47 (1983) 79–90.